Nonpolynomial Spline Method for Solving Linear Fractional Differential Equations

Nabaa N. Hasan and Ahmed J. Mohammad

Abstract---*The main objective of this paper define linear differential operator L of order n with its adjoint operator L* to solve L* L. = 0 to get non polynomial spline (generalized spline) basis. Different operators of order n, is used to get different basis of the method. "The efficiency of the method will be illustrated by solving test examples of linear fractional differential equations(LFDE) of order α", where 0 <α<1 and 1 <α< 2 in sense of Caputo definition is proposed.*

Keywords---*Non Polynomial Spline, Linear Fractional Differential Equation, Caputo Derivative.*

I. INTRODUCTION

Fractional calculus is a calculus of integrals and derivative of any arbitrary real or complex order, which is investigated during the past three decades in numerous areas of science and engineering, bioscience, control system and so on, [Guo, 2011]. Several methods for approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. Some of these methods Adomain decomposition method, Homotopy analysis method, iteration method, variational iteration method is applied for finding the solution of differential equations [Iqbal, 2016],[Ramzi, 2016],[Qi, 2012].

A correction functional is constructed by general Lagrange multiplier, which can be identified by variational theory[Belal, 2009]. This technique provides a sequence of functions, which converges to the exact solution of the problem without discretization of the variables [Abbas, 2009]. In [Gupta, 2015] Laplace Homotopy analysis method, is applied to obtain the approximate solutions of fractional linear and non-linear differential equations . In [Roberto, 2018] illustrate the basic principles behind some methods for LFDE, thus to provide a short tutorial on the numerical solution of FDE . In [Nabba, 2009] generalization of generalized spline functions for two-dimensional spaces by tensor product method to get two-dimensional approximation.

II. GENERALIZED SPLINE APPROXIMATION

Consider the linear differential operator of order n , [Doaa, 2018]:

$$\mathrm{L} = \mathrm{a_n}(\mathrm{x})\mathrm{D^n} + \mathrm{a_{n-1}}(\mathrm{x})\mathrm{D^{n-1}} + \cdots + a_1(x)D + a_0(x) \qquad (1)$$

Defined on the interval [a, b], the coefficients $a_k(x)$ are of class C^n(a, b). Assume that $a_n(x) \neq 0$ on [a, b], here the formal adjoint $\mathrm{L^n}$ of L is given by:

$$\mathrm{L^n} = (-1)^{\mathrm{n}}\, \mathrm{D^n}\{\mathrm{a_n}(\mathrm{x})\} + (-1)^{\mathrm{n}-1}\, \mathrm{D^{n-1}}\{\mathrm{a_{n-1}}(\mathrm{x})\} + \cdots + (-1)\, \mathrm{D}\,\{\mathrm{a_1}(\mathrm{x})\} + \mathrm{a_0}(\mathrm{x}) \qquad (2)$$

Definition(1),[Joseph, 2009]: The function S:[a,b]→R is an interpolation generalized spline function of f associated with the partition Δ and the operator L, $S(x) = f(x)$ on Δ, and $S^{(k)}(x_i) = f^{(k)}(x_i)$, for $i = 0$, N and

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$k = 1,2, \dots, n-1$.

Remarks, [Doaa, 2018]:

1. The function $S_{\Delta}(x)$ is said to be a spline function with respect to the partitionΔ or a spline on Δ, interpolating to the values y_i at the partition location.
2. $S_{\Delta}(x)$ has an order 2n when the order of the operator L*L is to be indicated in defining $S_{\Delta}(x)$.
3. For L= D^n the solution of L*L $S_{\Delta}(x) = 0$ give a polynomial splines.

Some Properties of Caputo Fractional Operator

(1) $D_*^{\alpha} f(t) = J^{n-\alpha} D^n f(t)$. [Batool, 2019]

(2) $D_*^{\alpha}\big(\lambda f(t) + g(t)\big) = \lambda D_*^{\alpha} f(t) + D_*^{\alpha} g(t)$. [Batool, 2019]

(3) $D_*^{\alpha} c = 0$. [Batool, 2019]

(4) $D^{\alpha} D^{\beta} f = D^{\beta} D^{\alpha} f = D^{\alpha+\beta} f$. [Batool, 2019]

(5) The Caputo fractional derivative of the power function satisfies:

$$D^{\alpha} t^{\text{Ƥ}} = \begin{cases} \dfrac{\Gamma(\text{Ƥ}+1)}{\Gamma(\text{Ƥ}-\alpha+1)} t^{\text{Ƥ}-\alpha} & n-1 < \alpha < n, \text{Ƥ} > n-1, \text{Ƥ} \in R \\ 0 & n-1 < \alpha < n, \text{Ƥ} \le n-1, \text{Ƥ} \in R \end{cases}$$

For example: to evaluate $D^{1/3} t^3$

$$D^{\frac{1}{3}} t^3 = \frac{3}{\Gamma(\frac{11}{3})} t^{\frac{8}{3}} \quad , \quad 0 < \alpha < 1$$

In the following section generalized spline function is applied to find the approximate solution of the LFDE

III. SOLUTION OF LINEAR FRACTIONAL DIFFERENTIAL EQUATION

Consider the following Caputo fractional initial value problem(IVP):

$$\begin{gathered} D^{\alpha} y(t) = f(t, y(t)) \\ y(t_0) = y_0, y'(t_0) = y_0^{(1)}, \dots, y^{(m-1)}(t_0) = y_0^{(m-1)} \end{gathered} \tag{3}$$

where $f(t, y)$ is assumed to be continuous and $y_0, y_0^{(1)}, \dots, y^{(m-1)}$ are the assigned values of the derivatives at t_0 where m is integer, $m-1 \le \alpha \le m$

The generalized spline function is applied to find the approximate solution of the IVP is given in eq. (3).

Let S $(t) = \sum_{j=1}^{2n} c_j q_j(t)$, $0 \le t \le 1$(4)

where$q_j(t)$, $j = 1,2,3, \dots, 2n$ be the basis functions of generalized spline S(t), $2n$ is order of L*L and $c_1, c_2, \dots, c_{2n}$ are constants to be found

Subsisting (4) in (3) we obtain:-

$$D^{\alpha}\Big(\sum_{j=1}^{2n} c_j q_j(t)\Big) = f\Big(t, \sum_{j=1}^{2n} c_j q_j(t)\Big) \tag{5}$$

$$\sum_{j=1}^{2n} c_j D^{\alpha} q_j(t) - \sum_{j=1}^{2n} c_j q_j(t) = f(t) \tag{6}$$

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$$\sum_{j=1}^{2n} c_j \, [D^{\alpha} q_j(t) - q_j(t)\,] = f(t)$$

Let $A_j(t_i) = D^{\alpha} q_j(t_i) - q_j(t_i)$, $i = 1,2,3, \dots, N, \quad N \in \mathbb{R}$

Adding the initial conditions of eq. (3) as new rows in the following matrices:

$$A = \begin{bmatrix} A_1(t_0) & A_2(t_0) & \dots & A_{2n}(t_0) \\ \vdots & \vdots & & \vdots \\ A_1(t_N) & A_2(t_N) & \dots & A_{2n}(t_N) \\ y_1(0) & y_2(0) & \dots & y_{2n}(0) \\ y_1'(0) & y_2'(0) & \dots & y_{2n}'(0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(0) & y_2^{(n-1)}(0) & \cdots & y_{2n}^{(m-1)}(0) \end{bmatrix}, \text{C=} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2n} \end{bmatrix}, F = \begin{bmatrix} f(t_0) \\ \vdots \\ f(t_N) \\ y_0 \\ \vdots \\ y_0^{(m-1)} \end{bmatrix} \qquad (7)$$

Or in the system from:

$$AC = F \qquad (8)$$

A and F are constant matrices

Calculate, $C = A^{-1}F$ to find the c_j , $j = 1,2, \cdots, 2n$

and substitute this solution in eq. (4) to get the approximate solution of eq. (3).

Theorem (1), [Joseph, 2009], [Mariyah, 2005]: Let $\alpha \epsilon$R, n-1$< \alpha <$ n, n$\in N$, $\lambda \in$C.

Then the caputo fractional derivative of the exponential function has the form:

$$D^{\alpha} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda\, t)$$

where E_{α} , is the one-parameter function of Mittag-leffler type.

IV. Convergence Analysis

In this section we will illustrate converges of Caputo fractional differential equations by Picard's iterative method, [Rainey, 2017] [.

Definition(2): The Beta function, $\beta(x, y)$, is defined by

$$\beta(x, y) = \int_0^1 (1-s)^{x-1} s^{y-1} ds, where\ Re(x), Re(y) > 0.$$

It should also be known that the Beta and Gamma functions have the following relationship: $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Definition(3): The Riemann-Lioville integral of arbitrary order α defined by

$$D^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, 0 < \alpha < 1$$

Note that the definition of Riemann-Lioville integral of order α for $0 < \alpha < 1$, is the same as Caputo integral of order α.

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Consider Caputo fractional IVP:

$$\begin{cases} D^{\alpha} = f(t,u), \\ u(t_0) = u_0 \end{cases} (9)$$

where $0< \alpha \leq 1$, The integral representation of (9) is by

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s,u(s)\big)ds. (10)$$

In order to prove that the solution of the IVP(9) exists and is unique on some interval, it is enough to prove that the solution of the integral equation (10) exists and is unique.

Definition(4): A function, $f(t,u) \in C\big[[t_0, t_0+T] \times R, R\big]$, is said to be a Lipschitz function in u if for any u_1,u_2 there exists an $L> 0$ such that

$$|f(t,u_1) - f(t,u_2)| \leq L|u_1 - u_2|.$$

Definition(5): The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}\big(\lambda(t^{\alpha})\big) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^k}{\Gamma(k\alpha+\beta)}, \qquad (11)$$

where $\alpha, \beta > 0$, and λ is a constant.

In particular, if $\beta = 1$ in (11), then we have:

$$E_{\alpha,1}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^k}{\Gamma(k\alpha+1)}, \qquad (12)$$

Definition(6): Weiestrass*M*-Test. Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on some domain *D*. If there are constants $M_n > 0$ such that $|f_n(x)| < M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D.

Theorem(2): Consider the IVP (9), where *f* is continuous Lipschitizian function on the closed rectangle

$$R := \{(t,x) | t \in [t_0 - a, t_0 + a], x \in [u_0 - b, u_0 + b]\}.$$

Let M > 0 be such that $|f(t,u)| < M$. Then , the IVP (9) has a unique solution on $I = [t_0 + t_0 + h]$, where $0 <\alpha < 1$ and . Furthermore, the iterations

$$u_n(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f\big(s, u_{n-1}(s)\big)ds, \quad (13)$$

with the initial approximation being $u_0(t) = u_0$, converge uniformly to the solution to the IVP (9).

V. Illustrative Examples

To show the efficiency of generalized spline method, we solve three examples of linear fractional differential equations. The first example of order $\alpha = 1.5$ with linear differential operator$\mathrm{L} = D^2 + D - 6$. In the second example, we use $\mathrm{L} = D^2$ to get polynomial approximation. The third example $\alpha = 0.5$ with $\mathrm{L} = D^2 + D - 6$. To show the approach of the generalized spline method we will compare our results with results given in [Iqbal, 2016] [Ramzi, 2016].

Example (1): Consider the following linear fractional differential equation:

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$$D^{1.5}y(x) = x^{1.5}\, y(x) + 4\sqrt{\frac{t}{\pi}} - x^{3.5} \quad (20)$$

with the initial conditions $y(0) = 0$ (21)

The exact solution is $y(x) = x^2$

Let Δ be a partition such that; $\Delta: 0 = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = 1$

Where $h = 0.2$, then $x_0 = 0$, x_1 =0.2,$x_2 = 0.4$,$x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

Appling generalized spline function to fractional differential equation (20).

Let L be the differential operator of order 2, such that L $= D^2 + D - 6$

$L \; ^*L = D^4 - 13D^2 + 36$, with basis functions:

$$u_1(x) = e^{3x}, u_2(x) = e^{-3x}, u_3(x) = e^{2x}, u_4(x) = e^{-2x}$$

which gives the generalized spline function:

$$S(x) = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{2x} + c_4 e^{-2x} \quad (22)$$

Now substitutingeq. (22) in the initial conditioneq. (21) we get:

$$c_1 + c_2 + c_3 + c_4 = 0$$

Now for applying eq. (22) in eq. (20), and by system given in eq.(7) we get:

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 10.665 & -0.69 & 5.177 & -0.966 \\ 17.824 & 0.405 & 6.782 & -0.059 \\ 31.639 & 0.483 & 9.702 & 0.221 \\ 56.357 & -0.082 & 14.199 & 0.283 \\ 98.947 & -2.035 & 20.893 & 0.209 \end{bmatrix}, C=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \text{ and } F=\begin{bmatrix} 0 \\ 1.006 \\ 1.387 \\ 1.581 \\ 1.561 \\ 1.257 \end{bmatrix} \quad (23)$$

Finally, Gauss elimination method may be used to solve system(23) to find:

C_1=-2.824×10^{-3}, C_2= 0.714, C_3=0.15, C_4=0.831.

So the approximate solution is :

$$s(x) := -2.824\times 10^{-3}\cdot e^{3x} + 0.714e^{-3x} + 0.15e^{2x} - 0.831e^{-2x}$$

The approximation results is showen in table(1) and figure(1):

Table1:

X	Y(x)	S(x)	|Y(x)-S(x)|	App.[Iqbal, 2016]
0	0	0.030176	0.03	0
0.1	0.01	0.027977	0.018	0.009999
0.2	0.04	0.053443	0.13	0.039999
0.3	0.09	0.100600	0.011	0.089999
0.4	0.16	0.166115	6.115×10^{-3}	0.1599999
0.5	0.25	0.248693	1.307×10^{-3}	0.249999
0.6	0.36	0.348664	0.011	0.359999
0.7	0.49	0.467730	0.022	0.489999
0.8	0.64	0.608822	0.031	0.639999
0.9	0.81	0.776048	0.034	0.809999
1	1	0.975	0.025	0.999999

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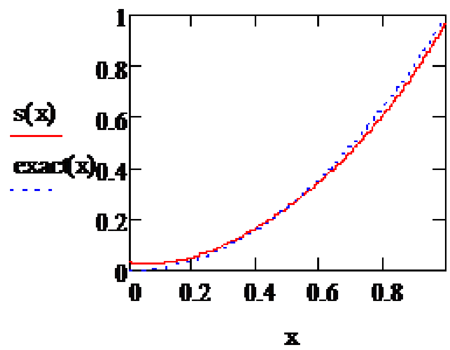

Figure 1: Exact and approximate solutions of example (1)

Example (2):Consider following linear fractional differential equation

$$D^{1.5}y(x) = x^{1.5}\,y(x) + 4\sqrt{\frac{t}{\pi}} - x^{3.5} \qquad (24)$$

with the initial conditions $y(0) = 0$ (25)

The exact solution is: $y(x) = x^2$ [De, 2017].

Let Δ be a partition such that; $\Delta: 0 = x_0 < x_1 < x_2 < x_3 = 1$

where $h = \frac{1}{3}$, then $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$

Appling generalized spline function to fractional differential equation (24).

Let$L = L *= D^2$, then $L * L = D^4$

We have the solutions: $u_1(x) = 1$, $u_2(x) = x$, $u_3(x) = \frac{x^2}{2}$, $u_4(x) = \frac{x^3}{6}$

which gives the generalized spline polynomial:

$$S(x) = c_1 + c_2 x + c_3\frac{x^2}{2} + c_4\frac{x^3}{6} \qquad (26)$$

The coefficients c_1 , c_2 , $c_3$and c_4 in eq. (25) are unknown four algebraic equations

Now substituting eq. (26) in the initial condition eq. (25) we have:

$$c_1 = 0 (27)$$

$$c_2 = 0 \ (28)$$

Now, for applying eq. (26) in eq. (24), we have:

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.658 & 0.913 & 0.641 & 0.144 \\ -1.063 & 0.328 & 0.8 & 0.383 \\ -1.282 & -0.436 & 0.628 & 0.586 \end{bmatrix},\ C=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},\ \text{and } F=\begin{bmatrix} 0 \\ 1.282 \\ 1.601 \\ 1.257 \end{bmatrix} \ (29)$$

Finally Gauss elimination method may be used to solve system (29) to find:

$C_1=0$, $C_2=-2.668\times10^{-3}$, $C_3=2.005$, $C_4=-5.73\times10^{-3}$

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So the approximate solution is:

$$S(x) = (-2.668\times10^{-3})x + (2.005)\ x^2/2 + (-5.73\times10^{-3})\ x^3/6$$

The approximation results is showen in table(2) and figure(2):

Table 2

X	Y(x)	S(x)	|Y(x)-S(x)|	App.[Iqbal,2016]
0	0	0	0	0
0.1	0.01	0.009757	2.428×10^{-4}	0.00999
0.2	0.04	0.039558	4.412×10^{-4}	0.03999
0.3	0.09	0.089398	6.012×10^{-4}	0.08999
0.4	0.16	0.159271	7.283×10^{-4}	0.15999
0.5	0.25	0.249171	8.284×10^{-4}	0.24999
0.6	0.36	0.359092	9.701×10^{-4}	0.35999
0.7	0.49	0.489029	9.702×10^{-4}	0.48999
0.8	0.64	0.638976	1.023×10^{-3}	0.63999
0.9	0.81	0.808927	1.072×10^{-3}	0.80999
1	1	0.998877	1.123×10^{-3}	0.99999

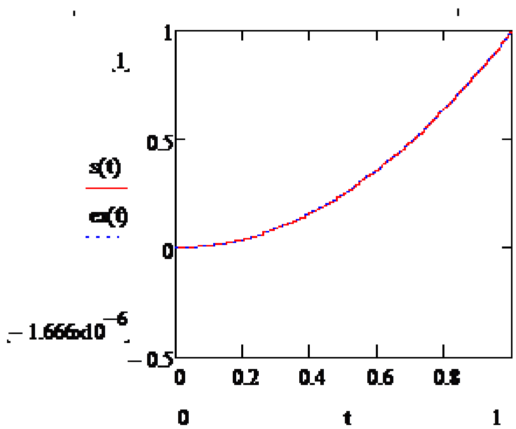

Figure 2: Exact ex(t) and approximate s(t) solutions of example (2)

Example (3): For the linear fractional differential equation:

$$D^{\alpha}\ y(x) = x^2 + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}\ y(x) \tag{30}$$

with the initial condition: $y(0) = 0$(31)

The exact solution to this problem is $y(x) = x^2$

Let Δ be a partition such that; $\Delta: 0 = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = 1$

where $h = 0.2$, then $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

Appling the generalized Spline function to the fractional differential eq.(30).

Let L be the differential operator of order 2, such that $L = D^2 + D - 6$

$L^*L = D^4 - 13D^2 + 36$, with basis functions:

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$$u_1(x) = e^{3x}, u_2(x) = e^{-3x}, u_3(x) = e^{2x}, u_4(x) = e^{-2x}$$

which gives the generalized spline function:

$$S(x) = c_1e^{3x} + c_2e^{-3x} + c_3e^{2x} + c_4e^{-2x} \tag{32}$$

Now replacing eq. (32) in the initial condition eq. (31) we get:

$$c_1 + c_2 + c_3 + c_4 = 0$$

Now for applying eq. (32) in eq. (31), and by system given in eq.(7) we get:

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4.116 & -0.483 & 2.819 & -0.108 \\ 8.373 & -0.729 & 4.725 & -0.413 \\ 15.922 & -0.758 & 7.446 & -0.54 \\ 29.571 & -0.718 & 11.442 & -0.584 \\ 54.357 & -0.65 & 17.363 & -0.587 \end{bmatrix},\ C=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},\ \text{and } F=\begin{bmatrix} 0 \\ 0.175 \\ 0.541 \\ 1.059 \\ 1.717 \\ 2.505 \end{bmatrix} \tag{33}$$

Finally, gauss elimination method may be used to solve system (32) to find:

C_1=-0.015,C_2=0.874, C_3=0.188, C_4=-1.044

So the approximate solutions S(x) is:

$$S(x) = -0.015\, e^{3x} + 0.874\, e^{-3x} + 0.188\, e^{2x} - 1.044\, e^{-2x}$$

The approximation results is showen in table(3) and figure(3):

Table 3

X	Y(x)	S(x)	|Y(x)-S(x)|	App.[Ramzi, 2016]
0	0	0.003	3×10^{-3}	0
0.1	0.01	0.002096	7.904×10^{-3}	0.010116
0.2	0.04	0.032978	7.021×10^{-3}	0.040156
0.3	0.09	0.088046	1.953×10^{-3}	0.090181
0.4	0.16	0.162744	2.744×10^{-3}	0.160200
0.5	0.25	0.254761	4.761×10^{-3}	0.250215
0.6	0.36	0.363461	3.462×10^{-3}	0.360227
0.7	0.49	0.489464	5.353×10^{-3}	0.490237
0.8	0.64	0.634329	5.67×10^{-3}	0.640246
0.9	0.81	0.800303	9.69×10^{-3}	0.810254
1	1	-1.044	9.917×10^{-3}	1.000261

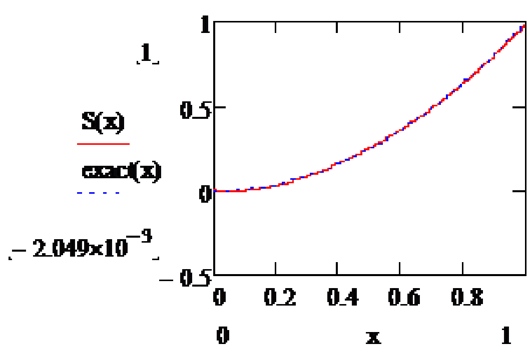

Figure 3: Exact exact(x) and approximate S(x) solutions of example (3)

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VI. CONCLUSION

We can conclude that non-polynomial spline (generalized spline) functions have been proved powerful method for solving linear differential equations of fractional order. Different kinds of linear differential operator are proposed to get basis of generalized spline functions, which are shown, in three examples. The results compare with the exact solution and method given in [Iqbal, 2016] [Ramzi, 2016]. Mathcad 15 is used for computation.

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