An Optimal Ninth Order Iterative Method for Solving Non-Linear Equations

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Abstract--- *In this paper, we suggest and analyze a new ninth order three step iterative method for solving non linear equations. This method is compared with the existing ones through some numerical examples to exhibit its superiority.*

Keywords--- *Iterative Method, Nonlinear Equation, Newton's Method, Convergence Analysis.*

I. INTRODUCTION

Iterative methods are widely used for finding roots of a nonlinear equation of the form

$$f(x)=0 \tag{1.1}$$

Where $f:D\subset R\to R$ is a scalar function on an open interval D and $f(x)$ may be algebraic, transcendental or combined of both. In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations, see [1-7] and the references therein.

The most widely used algorithm for solving "(1.1)" by the use of value of the function and its derivative is the well known quadratic convergent Newton's method (NM) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n=0,1,2,...) \tag{1.2}$$

starting with an initial guess x_0 which is in the vicinity of the exact root x^*. The efficiency index of (1.2) is $\sqrt{2} = 1.4142$.

A fourth order three step Iterative method for solving Nonlinear Equations Using Decomposition method (AAU) suggested by Ahmad [2] is given by

For a given x_0 , we compute x_{n+1} by using

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$$\left.\begin{aligned} z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ y_n &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n)} \\ x_{n+1} &= y_n - \frac{2f(y_n)}{f'(x_n)+f'(y_n)} \\ & \qquad (n=0,1,2,\ldots) \end{aligned}\right\} \tag{1.3}$$

This method has forth order convergence and efficiency index is $\sqrt[5]{4}=1.319$.

A third order three step iterative methods for nonlinear equations (NR) suggested by Aslam Noor [5] is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -\frac{f(y_n)}{f'(x_n)} \\ x_{n+1} &= x_n \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n+z_n)}{f'(x_n)} \\ & \qquad (\mathrm{n}=0,1,2,\ldots) \end{aligned}\right\} \tag{1.4}$$

This method has third order convergence and efficiency index is $\sqrt[4]{3}=1.31$.

In section II, we develop a new three step iterative method and their convergence criterion is discussed in section III. Few numerical examples are considered to show the superiority of this method in the concluding section.

II. NINTH ORDER CONVERGENT METHOD

Consider x^* be the exact root of "(1.1)" in an open interval D in which $f(x)$ is continuous and has well defined first and second derivatives. Let x_n be the n^{th} approximate to the exact root x^* of "(1.1)" and

$$x^* = x_n + e_n \tag{2.1}$$

where e_n is the error at the n^{th} stage.

Therefore, we have

$$f\left(x^*\right)=0 \tag{2.2}$$

Expanding $f\left(x^*\right)$ by Taylor's series about x_n, we have

$$f\left(x^*\right) = f(x_n) + \left(x^*-x_n\right) f'(x_n) + \frac{\left(x^*-x_n\right)^2}{2!} f''(x_n) + \ldots \tag{2.3}$$

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$$f\left(x^*\right) = f\left(x_n\right) + e_n f'\left(x_n\right) + \frac{e_n^2}{2} f''\left(x_n\right) + \ldots \tag{2.4}$$

Assuming e_n is small enough and neglecting higher powers of e_n starting from e_n^3 onwards, we obtain from "(2.2)" and "(2.4)" as

$$e_n^2 f''(x_n) + 2e_n f'(x_n) + 2f(x_n) = 0$$

$$e_n = -\frac{2f(x_n)}{f'(x_n)}\left[\frac{1}{1+\sqrt{1-2\rho}}\right] \tag{2.5}$$

where $$\rho_n = \frac{2f\left(y_n - \frac{f(y_n)}{f'(y_n)}\right)}{f(y_n)} \tag{2.6}$$

Replacing x^* by x_{n+1} in "(2.1)" and from "(2.5)" and "(2.6)" , we obtain

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} \cdot \frac{1}{1+\sqrt{1-\frac{4f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f(x_n)}}} \tag{2.7}$$

This scheme allows us to propose the following algorithm with the method "(1.1)" as first step, second formula in "(1.3)" as a second step and the equation "(2.7)" as the third step.

Algorithm

For a given x_0 , compute x_{n+1} by the iterative schemes

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.8}$$

$$y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(w_n)} \tag{2.9}$$

$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(y_n)}\left(\frac{1}{1+\sqrt{1-2\rho_n}}\right) \tag{2.10}$$

where $$\rho_n = \frac{2f\left(y_n - \frac{f(y_n)}{f'(y_n)}\right)}{f(y_n)} \tag{2.11}$$

This algorithm "(2.1)" requires 3 functional evaluations, 3 of its first derivatives and can be called as Ninth Order Convergent Method (NOCM).

III. CONVERGENCE CRITERIA

Thoerem: Let $x^* \in D$ be a single Zero of a sufficiently differentiable function $f : D \subset R \to R$ for an open interval

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D . If x_0 is in the vicinity of x^*.Then the algorithm has ninth order convergence.

Proof: Let x^* be a single zero of $f(x)=0$ and $x^*=x_n+e_n$,

then $f\left(x^*\right)=0$ .If x_n be the n[th] approximate to the root of $f(x)=0$, Then expanding g $f\left(x_n\right)$ about Using Taylor's expansion,

$$\text{we have } f(x_n) = f(x^*) + e_n f'(x^*) + \frac{e_n^2}{2!} f''(x) + \frac{e_n^3}{3!} f'''(x) + \dots$$

$$f(x_n) = f'(x^*)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + \dots\right] \tag{3.1}$$

$$f'\left(x_n\right) = f'\left(x^*\right)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + \dots\right] \tag{3.2}$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - \left(2c_3 - 2c_2^2\right)e_n^3 - \dots \tag{3.3}$$

Substituting"(3.3)" in "(2.8)" , we get

$$w_n = x^* + c_2 e_n^2 + \left(2c_3 + 2c_2^2\right)e_n^3 + \dots \tag{3.4}$$

$$f(w_n) = f'\left(x^*\right)\left[c_2 e_n^2 + \left(2c_3 + 2c_2^2\right)e_n^3 + \dots\right]$$

$$f'(w_n) = f'\left(x^*\right)\left[1 + 2c_2^2 e_n^2 + \left(4c_2 c_3 - 4c_2^3\right)e_n^3 + \dots\right] \tag{3.5}$$

Replacing"(3.1)", "(3.2)" and "(3.3)" in "(2.9)", we obtain

$$y_n = x^* + \left(\frac{1}{2}c_3 + c_2^2\right)e_n^3 + \left(c_4 + \frac{3}{2}c_2 c_3 - 3c_2^3\right)e_n^4 + \dots \tag{3.6}$$

$$= x^* + \omega$$

$$\text{where } \quad \omega = \left(\frac{1}{2}c_3 + c_2^2\right)e_n^3 + \left(c_4 + \frac{3}{2}c_2 c_3 - 3c_2^3\right)e_n^4 + o\left(e_n^4\right)$$

$$f(y_n) = f'(x^*)\left[\omega + c_2\omega^2 + c_3\omega^3 + \dots\right] \tag{3.7}$$

$$f'\left(y_n\right) = f'\left(x^*\right)\left[1 + 2c_2\omega + 3c_3\omega^2 + \dots\right] \tag{3.8}$$

Now

$$y_n - \frac{f(y_n)}{f'(y_n)} = x^* + c_2\omega^2 + \left(2c_3 - 2c_2^2\right)\omega^3 + \left(3c_4 - 7c_2 c_3 + 4c_2^3\right)\omega^4 + o\left(\omega^5\right) \tag{3.9}$$

$$y_n = x^* + R$$

$$\text{where , } R = c_2\omega^2 + \left(2c_3 - 2c_2^2\right)\omega^3 + o\left(\omega^4\right)$$

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$$f\left(y_n - \frac{f(y_n)}{f'(y_n)}\right) = f'\left(x^*\right)\left[R + c_2R^2 + c_3R^3 + \ldots\right] \quad (3.10)$$

Putting “(3.7)”, “(3.10)” in “(2.11)” , we have

$$\rho_n = p_1\omega + p_2\omega^2 + p_3\omega^3 + \ldots \quad (3.11)$$

where $p_1 = 2c_2, p_2 = -4c_3 + 6c_2^2, p_3 = 2c_2^2 - 4c_2c_3 + 6c_2^3$

$$\text{Now } \sqrt{1-2\rho_n} = 1 - p_1\omega + \left(\frac{-p_1^2}{2} - p_2\right)\omega^2 + o(\omega_3)$$

$$1 + \sqrt{1-2\rho_n} = 2\left[1 + M_1\omega + M_2\omega^2 + M_3\omega^3 + \ldots\right] \quad (3.12)$$

$$\text{where, } M_1 = \frac{-p_1}{2}, M_2 = -\left(\frac{p_1^2}{4} + \frac{p_2^2}{2}\right), \ldots$$

$$\left(1+\sqrt{1-2\rho_n}\right)^{-1} = \frac{1}{2}\left[1 + c_2\omega + \left(2c_2^2 + 8c_3^2 + 18c_2^4 - 24c_3c_2^2\right)\omega^2 + o\left(\omega^3\right)\right] \quad (3.13)$$

Now

$$\frac{2f(y_n)}{f'(y_n)}\left[\frac{1}{1+\sqrt{1-2\rho_n}}\right] = \left[\omega + \left(3c_2^2 + 8c_3^2 + 18c_2^4 - 24c_3c_2^2 - 2c_3\right)\omega^3 + \ldots\right] \quad (3.14)$$

$$= x^* + T$$

$$\text{where, } T = \left(3c_2^2 + 8c_3^2 + 18c_2^4 - 24c_3c_2^2 - 2c_3\right)\omega^3 + o\left(\omega^4\right)$$

Substituting “(3.14)” in “(2.10)” , we get

$$\mathrm{x}_{n+1} = \left(3c_2^2 + 8c_3^2 + 18c_2^4 - 24c_3c_2^2 - 2c_3\right)\left(\left(\frac{1}{2}c_3 + c_2^2\right)^3 e_n^9\right) + \ldots$$

Therefore, this new method called NOCM has ninth order convergence and its efficiency index is $\sqrt[6]{9} = 1.442$

IV. Numerical Examples

We consider the same examples considered by Vatti. et. al [8] and compared NOCM with NM, AAU and NR methods. The computations are carried out by using mpmath-PYTHON software programming and comparison of number of iterations for these methods are obtained such that $|x_{n+1} - x_n| < 10^{-201}$ and $|f(x_{n+1})| < 10^{-201}$.

The test functions and simple zeros are given below $f_1(x) = \sin^2 x - x^2 + 1$,

$$x^* = 1.40449164$$

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$$f_2(x) = x^2 - e^x - 3x + 2\,,\; x^* = 0.25753028$$

$$f_3(x) = \cos x - x\,,\; x^* = 0.73908513$$

$$f_4(x) = (x-1)^3 - 1\,,\; x^* = 0$$

$$f_5(x) = x^3 - 10\,,\; x^* = 2.1544346$$

Table IV (A): Comparison of Efficiency Index

Methods	P	N	EI
NM	2	2	1.414
NR	2	4	1.3
AAU	4	5	1.319
NOCM	9	6	1.442

where P is order of the convergence, N is the number of functional values per iteration and EI is the Efficiency Index.

Table IV (b): Comparison of Different Methods

f	Method	x_0	*n*	*er*	*fv*
f_1	NM	-1	9	3.1(200)	7.7(200)
	NR		11	6.0(200)	7.7(200)
	AAU		6	3.7(200)	7.7(200)
	NOCM		4	6.0(200)	7.7(200)
f_2	NM	2	9	8.2(202)	3.3(200)
	NR		9	1.6(201)	3.3(201)
	AAU		6	1.6(201)	3.3(201)
	NOCM		4	3.3(201)	1.6(201)
f_3	NM	1.7	9	1.6(201)	2.4(201)
	NR		9	3.3 (201)	2.4(201)
	AAU		5	3.3(201)	2.4(201)
	NOCM		4	6.5(201)	1.4(200)
f_4	NM	2.5	10	0	0
	NR		10	0	0
	AAU		7	0	0
	NOCM		5	0	0
f_5	NM	2	9	1.6(200)	2.1(199)
	NR		9	6.9(200)	1.2(199)
	AAU		6	3.3(200)	2.1(199)
	NOCM		4	3.3(200)	2.1(199)

Where x_0 is the initial approximation, *n* is the number of iterations, *er* is the error and *fv* is the functional value.

V. CONCLUSION

The above computational results exhibit the superiority of the new method NOCM over the Newton's method (NM) and the methods NR, AAU in terms of number of iterations and accuracy.

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