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Common Fixed Point Theorems For Occasionally Weakly Compatible Mappings In Semi-Metric Space

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Abstract
In this paper, we establish a common fixed point theorem for three pairs of self mappings in semi-metric space for occasionally weakly compatible mappings which improves and extends similar known results in the literature.

Keywords: *Semi-metric space, occasionally weakly compatible mappings, common fixed point.*

AMS Mathematics Subject Classification (2010): 54 H 25, 47 H 10

1. Introduction

The fixed point theory has become a part of non-linear functional analysis since 1960. It serves as an essential tool for various branches of mathematical analysis and its applications. Polish mathematician Banach published his contraction Principle in1922. In 1928, Menger[16] introduced semi-metric space as a generalization of metric space. In 1976, Cicchese [6] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces. In 1986, Jungck [12] introduced the notion of compatible mappings. In 1997, Hicks and Rhoades[8] generalized Banach contraction principle in semi-metric space. In 1998, Jungck and Rhoades [13] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. Recently in 2006,Jungck and Rhoades [14] introduced occasionally weakly compatible mappings which is more general among the commutativity concepts. Jungck and Rhoades[14] obtained several common fixed point theorems using the idea of occasionally weakly compatible mappings. Several interesting and elegant results have been obtained by various authorsin this direction. There have been interesting generalized and formulated results in semi- metric space initiated by Frechet [7], Menger [16] and Wilson[18]. Also, in this paper, we prove a common fixed point theorem for three pairs of self-mappings using occasionally weakly compatible mappings.
Let X be a non-empty set and $d: X \times X \to [0, \infty)$, Then, (X, d) is said to be a semi-metric space (symmetric space) if and only if it satisfies the following:
W1 : $d(x, y) = 0$ if and only if $x = y$, and
W2 : $d(x, y) = d(y, x)$ if and only if $x = y$ for any $x, y \in$ X.

The difference of a semi-metric and a metric comes from the triangle inequality.

Definition 1.1. [1] Let A and B be two self-mappings of a semi-metric space (X, d).Then, A and B are said to be **compatible** if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 1.3. [1] Let A and B be two self-mappings of a semi-metric space (X, d). Then, A and B are said to be **weakly compatible** if they commute at their coincidence points.

Definition 1.4. [14] Let A and B be two self-mappings of a semi-metric space (X, d). Then, A and B are said to be **occasionally weakly compatible (owc)** if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.

Example 1.1. Let us consider X = [2, 20] with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define a self map A and B by

$$A(2) = 2atx \text{ and } A(x) = 6 \text{ for } x > 2$$
$$B(2) = 2at = 2,\ B(x) = 12 \text{ for } 2 < x \leq 5 \text{ and } B(x) = x - 3 \text{ for } x > 5.$$

Now, $A(9) = B(9) = 6$, besides $x = 2$, $x = 9$ is another coincidence point of A and B.
$AB(2) = BA(2)$ but (9) = 6, $BA(9) = 3$, $AB(9) \neq BA(9)$. Therefore A and B are owc but not weakly compatible. Hence

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weakly compatible mappings are owc but not conversely.

Lemma 1.1. [14] Let (X, d) be a semi-metric space. If the self mappings A and B on X have a unique point of coincidence *w*=Ax=Bx, then *w* is the unique common fixed point of A and B.
In order to establish our result, we consider a function $\emptyset: R^+ \to R^+$ satisfying$(\emptyset 1) 0 < \emptyset(t) < t$, for $t > 0$, and $(\emptyset 2)$ for each $t > 0$, $\lim_{n\to\infty} \emptyset^n(t) = 0$.

2. Main Results

Theorem 2.1. Let (X, d) be a semi-metric space. Let *A*, *B*, *T*, *S*, *P* and *Q* be self-mappingsof *X* such that
(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (owc),
(ii) $d(ABx, TSy) \le \emptyset\left(\max\left\{d(Px, Qy).\frac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \frac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\right\}\right)$ for all $(x, y) \in X \times X$,

Then *AB*, *TS*, *P* and *Q* have a unique common fixed point. Furthermore, if the pairs (*A*, *B*) and (*T*, *S*) are commuting pair of mappings then *A*, *B*, *T*, *S*, *P* and *Q* have a unique common fixed point.

Proof: Since $\{AB, P\}$ and $\{TS, Q\}$ are owc, then there exists $x, y \in X$ such that $ABx = Px$ and $TSy = Qy$. We claim that $ABx = TSy$. Using condition (ii), we get

$$\begin{aligned} d(ABx, TSy) &\le \emptyset\left(\max\left\{d(Px, Qy), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \frac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\right\}\right) \\ &= \emptyset\left(\max\left\{d(ABx, TSy), \frac{1}{2}[d(ABx, ABx) + d(TSy, TSy)], \frac{1}{2}[d(ABx, TSy) + d(TSy, ABx)]\right\}\right) \\ &= \emptyset(\max\{d(ABx, TSy), 0, d(ABx, TSy)\}) \\ &= \emptyset(\max\{d(ABx, TSy)\}) \\ &= \emptyset\big(d(ABx, TSy)\big) \\ &< d(ABx, TSy) \end{aligned}$$

which is contradiction. So, $ABx = TSy$. Therefore, $ABx = Px = TSy = Qy$. … (2.1)
Moreover, if there is another point of coincidence z such that $ABz = Pz$, then using condition (ii), we get

$$ABz = Pz = TSy = Qy \ldots (2.2)$$

Also from (2.1) and (2.2), it follows that $ABz = ABx$. This implies that $z = x$.Hence, w = $ABx = Px$, for w ∈ X, is the unique point of coincidence of AB and P. ByLemma 1.1, w is the unique common fixed point of $AB$and P. Hence ABw = Pw = w. Similarly, there is a unique common fixed point$u$ ∈ X such that $u = TSu = Qu$. Suppose that $u \ne$ w. Then using condition (ii), we get.

$$\begin{aligned} d(w, u) &= d(ABw, TSu) \\ &\le \emptyset\left(\max\left\{d(Pw, Qu), \frac{1}{2}[d(ABw, Pw) + d(TSu, Qu)], \frac{1}{2}[d(ABw, Qu) + d(TSu, Pw)]\right\}\right) \\ &= \emptyset\left(\max\left\{d(w, u), \frac{1}{2}[d(w, w) + d(u, u)], \frac{1}{2}[d(w, u) + d(u, w)]\right\}\right) \\ &= \emptyset(\max\{d(w, u), 0, d(w, u)\}) \\ &= \emptyset\big(d(w, u)\big) \\ &< d(w, u) \end{aligned}$$

This is contradiction. Therefore, we have w = u. Hence, w is the unique common fixed point of *AB*, *TS*, *P* and Q. Finally, we need to show thatw is only the common fixed point of mappings *A*, *B*, *T*, *S*, *P* and *Q*. If the pairs (*A*, *B*) and (*T*, *S*) are commuting pairs, then for this, we can write Aw = $A(AB$w$)$ = $A(BA$w$)$ = $AB(A$w$)$. This implies that Aw=w. Also, Bw=$B(AB$w$)$= BA, $(B$w$)$=$AB(B$w$)$. This implies that Bw=w. Similarly, we have Tw=w and Sw=w.
Hence *A*, *B*, *T*, *S*, *P* and Q have a unique common fixed point.

Example 2.1. Consider $X = [0,1]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define selfmappings *A*, *B*, *T*, *S*, *P* and *Q* as $Ax = \frac{x+1}{2}$, $Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $S(x) = \frac{x+3}{4}$, $P(x) = \frac{3x+1}{4}$ and $Q(x) = \frac{2x+3}{5}$.
Also, the mappings satisfy all the conditions of above Theorem 2.1 and hence have a unique common fixed point x = 1.
On the basis of above Theorem 2.1, we have the following corollary.

Corollary 2.1. Let (X, d) be a semi-metric space. Let A, B, T, S, P and Q be self-mappings of X such that
(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (owc),
(ii) $d(ABx, TSy) \leq \emptyset\left(\max\left\{d(Px, Qy), d(ABx, Qy), d(TSy, Px), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)]\right\}\right)$ for all $(x, y) \in X \times X$ then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A, B) and (T, S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

In the above Theorem 2.1, if we take $A = B$ and $= S$, then we have the following corollary. This is the result of G. Jungck and B.E. Rhoades [13].

Corollary 2.2. Let (X, d) be a semi-metric space. Let A, T, P and Q be self- mappings of X such that
(i) $\{A, P\}$ and $\{T, Q\}$ are occasionally weakly compatible (owc),
(ii) $d(Ax, Ty) \leq \emptyset\left(\max\left\{d(Px, Qy), \frac{1}{2}[d(Ax, Px) + d(Ty, Qy)], \frac{1}{2}[d(Ax, Qy) + d(Ty, Px)]\right\}\right)$ for all $(x, y) \in X \times X$, then A, T, P and Q have a unique common fixed point.
In Theorem 2.1, if we take $A = B = Q$ and $T = S = P$, then we have the followingcorollary.

Corollary 2.3. Let (X, d) be a semi-metric space. Let A and T be self- mappings of X suchthat
(i) A and $T$are occasionally weakly compatible (owc),
(ii) $d(Ax, Ty) \leq \emptyset\left(\max\left\{d(Tx, Ay), \frac{1}{2}[d(Ax, Tx) + d(Ty, Ay)], \frac{1}{2}[d(Ax, Ay) + d(Ty, Tx)]\right\}\right)$ for all $(x, y) \in X \times X$, then A and T have a unique common fixed point.

Remarks 2.1. Our result generalizes the result of Jungck and Rhoades [14], Manro [15], and other similar results in the semi-metric space.

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