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Common Fixed Point Theorems For Compatible Mapping Of Type (A) Involving Rational Contractive Condition

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Abstract
Sessa [11], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [11] introduced the notion of weak commutativity. Motivated by Sessa [11], Jungck [6] generalized the concept of weak commuting by defining the term compatible mappings and proved that the weakly commuting mappings are compatible but the converse is not true. In recent years, several authors have obtained coincidence point results for various classes of mappings on a complete metric space utilizing these concepts. In this paper, we prove a common fixed point theorems for six mappings using the concept of compatible mapping of type (A). Our work generalizes some earlier results of Fisher [1], Jeong-Rhoades [4], Kannan [9] and others.

Keywords: - *Complete metric spaces, fixed points, compatible mapping, weak compatible mapping, compatible mapping of type (A).*

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1. INTRODUCTION AND PRELIMINARIES:

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's theorem and several others. Foremost among them is perhaps the weak commutativity condition introduced by Sessa [11] which can be described as follows:

1.1 Definition:
Let S and T be mappings of a metric space (X, d) into itself. Then (S, T) is said to be **weakly commuting** pair if

$$d(STx, TSx) \le d(Tx, Sx) \text{ for all } x \in X .$$

obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

1.2 Example:

Consider the set $X = [0, 1]$ with the usual metric. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$ for every $x \in X$. Then for all $x \in X$

$$STx = \frac{x}{4+2x}, \quad TSx = \frac{x}{4+x}$$

hence $ST \neq TS$. Thus S and T do not commute.
Again

$$d(STx, TSx) = \left| \frac{x}{4+2x} - \frac{x}{4+x} \right| = \frac{x^2}{(4+x)(4+2x)}$$

$$\le \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and so S and T commute weakly.

Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.
Jungck [6] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungek [6].

1.3 Definition:
Two self mappings S and T of a metric space (X, d) are **compatible** if and only if.

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$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in *X*. such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly any weakly commuting pair {*S*,*T*} in compatible but the converse need not be true as can be seen in the following example.

1.4 Example:

Let $Sx = x^3$ and $Tx = 2x^3$ with *X* = *R* with the usual metric. Then *S* and *T* are compatible, since

$|Tx - Sx| = |x^3| \to 0$ if and only if

$|STx - TSx| = 6|x^9| \to 0$ but

$|STx - TSx| \le |Tx - Sx|$ is not true for all $x \in X$, say for example at *x* = 1.

More, recently, Jungck et. al. [8] introduced the concept of compatible mapping of type (A) which is stated as follows:

1.5 Definition:

Let *S* and *T* be mappings from a metric space (*X, d*) into itself. The pair (*S, T*) is said to be **compatible of type (*A*)** on *X* if whenever $\{x_n\}$ is a sequence in *X* such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ in } X \text{ then}$$
$$d(STx_n, TTx_n) \to 0 \text{ and } d(TSx_n, SSx_n) \to 0 \quad as \quad n \to \infty$$

It is shown in [9] that under certain conditions the compatible and compatible (type A) mappings are equivalent for instance.

1.6 Proposition:

Let *S* and *T* be continuous self mapping on *X*. Then the pair (*S*, *T*) is compatible on X. where as in (Jungck [8], Gajic [2]) demonstrated by suitable examples that if *S* and *T* are discontinuous then the two concepts are independent of each other. The following examples also support this observation.

1.7 Example:

Let *X* = *R* with the usual metric we define *S, T* : $X \to X$ as follows.

$$Sx = \begin{cases} 1/x^2 & x \ne 0 \\ 0 & x = 0 \end{cases} \quad and \quad Tx = \begin{cases} 1/x^3 & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Both *S* and *T* are discontinuous at *x* = 0 and for any sequence $\{x_n\}$ in *X*, we have $d(STx_n, TSx_n) = 0$. Hence the pair (S,T) is compatible. Now consider the sequence $x_n = n \in N$. Then $Sx_n \to 0$ and $Tx_n \to 0$ as $n \to \infty$ and

$$d(STx_n, TTx_n) = |x^6 - x^9| \to \infty \quad as\ n \to \infty$$

Hence the pair (*S, T*) is not compatible of type (*A*)

1.8 Example:

Now we define

$$Sx = \begin{cases} 1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases} \quad and \quad Tx = \begin{cases} -1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

observe that the restriction of *S* and *T* on $(-\infty,1]$ are equal, thus we take a sequence $\{x_n\}$ in $(1,\infty)$. Then $\{Sx_n\} \subset (0,1)\ and\ \{Tx_n\} \subset (-1,0)$. Thus for every *n*, $TTx_n = 0$, $TSx_n = 1$, $STx_n = 0$, $SSx_n = 1$. So that $d(STx_n, TTx_n) = 0$, $d(TSx_n, TTx_n) = 0$ for every $n \in N$. This shows that the pair (*S, T*) is compatible of type (A). Now let $x_n = n$, $n \in N$. Then $Tx_n \to 0,\ Sx_n \to 0 \quad as\ n \to \infty\ and\ STx_n = 0,\ TSx_n = 1$ for every $n \in N$ and so $d(STx_n, TSx_n) \ne 0$ as $n \to \infty$ hence the pair (S, T) is not compatible.

Very recently concept of **weakly compatible** obtained by Jungck-Rhoades [6] stated as the pair of mappings is said to be weakly compatible if they commute at their coincidence point.

1.9 Example:
Let X = [2, 20] with usual metric define

$$Tx = \begin{cases} 2 & if \;\; x = 2 \\ 12 + x & if \;\; 2 < x \le 5 \\ x - 3 & if \;\; 5 < x \le 20 \end{cases} \quad and \quad Sx = \begin{cases} 2 & if \;\; x \in \{2\} \cup (5,20] \\ 8 & if \;\; 2 < x \le 5 \end{cases}$$

S and *T* are weakly compatible mappings which is not compatible. To see that *S* and *T* are not compatible of Type (A). Let us consider a decreasing sequence $\{x_n\}$ such that

$\lim x_n = 5$

Then

$$Tx_n = x_n - 3 \to 2; \; Sx_n = 2; \; STx_n = S(x_n - 3) = 8 \; and$$
$$TTxn = T(x_n - 3) = 12 + x_n - 3 \to 14, that\ is$$
$$\lim d(STx_n, TTx_n) = 6 \ne 0$$

and hence *S* and *T* are not compatible of type (A).

2. MAIN RESULT

The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [5].

2.1 Lemma:
Let $\{y_n\}$ be a sequence in a complete metric space (*X*, *d*). If there exists a $k \in (0,1)$ such that $d(y_{n+1}, y_n) \le k\ (y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in *X*.

2.2 Theorem:
Let *A,B,S,T, I* and *J* be self mappings of a complete metric space (*X*,*d*) satisfying $AB(X) \subset J(X), ST(X) \subset I(X),$ and for each $x, y \in X \;\; \alpha, \beta \ge 0, \alpha + \beta \le 1$
either

$$d(ABx, STy) \le \alpha \left[\frac{d(Ix, ABx).d(Ix, STy) + d(Jy, STy).d(Jy, ABx)}{d(Ix, ABx) + d(Jy, STy)} \right] + \beta d(Ix, Jy) \quad ....(1)$$

whenever $d(Ix, ABx) + d(Jy, STy) \ne 0$ *or*

$$d(ABx, STy) = 0 \text{ whenever } d(Ix, ABx) + d(Jy, STy) = 0 \quad ....(2)$$

(*a*) (*AB*, *I*) and (*ST*,*J*) are compatible of type (*A*), if one of *AB, ST*, *I* and *J* is continuous then *AB, ST*, *I* and *J*, have a unique common fixed point. Furthermore if the pairs (*A*,*B*), (*A*,*I*), (*B*,*I*) (*S*,*T*), (*S*,*J*) and (*T*,*J*) are commuting mappings then *A*,*B*,*S*,*T*,*I* and *J* have a unique common fixed point.

Proof:
We construct the sequence as follows. Let x_0 be an arbitrary point in *X*. Since $AB(X) \subset J(X)$ we can choose a point x_1 in *X* such that $ABx_0 = Jx_1$ again since $ST(X) \subset I(X)$ we can choose a point x_2 in *X* such that $STx_1 = Ix_2$. Using this argument repeatedly one can construct a sequence $\{z_n\}$ such that $z_{2n} = ABx_{2n} = Jx_{2n+1}$, $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$
for n = 0,1,2,..... brevity
Let us denote

$$U_{2n} = d(ABx_{2n}, STx_{2n+1})$$
$$U_{2n+1} = d(STx_{2n+1}, ABx_{2n+2}) \quad for \; n = 0,1,2,....$$

we distinguish two cases
Case I:
Suppose that $U_{2n} + U_{2n+1} \ne 0$ for n = 0,1,2....

Then on using inequality (2) we have

$$U_{2n+1} = d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$

$$\leq \alpha\left[\frac{d(Ix_{2n+2}, AB_{2n+2}).d(Ix_{2n+2}, ST_{2n+1}) + d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})}{d(Ix_{2n+2}, ABx_{2n+2}) + d(Jx_{2n+1}, STx_{2n+1})}\right] + \beta d(Ix_{2n+2}, Jx_{2n+1})$$

$$\leq \alpha\left[\frac{d(z_{2n+1}, z_{2n+2}).d(z_{2n+1}, z_{2n+1}) + d(z_{2n}, z_{2n+1}).d(z_{2n}, z_{2n+2})}{d(z_{2n+1}, z_{2n+2}) + d(z_{2n}, z_{2n+1})}\right] + \beta d(z_{2n+1}, z_{2n})$$

$$\leq \alpha\left[\frac{d(z_{2n+1}, z_{2n+2}).0 + d(z_{2n}, z_{2n+1}).d(z_{2n}, z_{2n+2})}{d(z_{2n}, z_{2n+2})}\right] + \beta d(z_{2n+1}, z_{2n})$$

$$d(z_{2n+1}, z_{2n+2}) \leq \alpha d(z_{2n}, z_{2n+1}) + \beta d(z_{2n}, z_{2n+1})$$

$$d(z_{2n+1}, z_{2n+2}) \leq (\alpha+\beta) d(z_{2n}, z_{2n+1})$$

Similarly one can show that

$$d(z_{2n}, z_{2n+1}) \leq (\alpha+\beta) d(z_{2n-1}, z_{2n})$$

Thus for every *n* we have

$$d(z_n, z_{n+1}) \leq k d(z_{n-1}, z_n), \quad ....(3)$$

where $k = \alpha + \beta < 1$. Therefore by Lemma 2.1, $\{z_n\}$ converges to some point $z \in X$.
Hence the sequence $\{ABx_{2n}\} = \{Jx_{2n+1}\}$ and $\{STx_{2n+1}\} = \{Ix_{2n+2}\}$
which are the subsequences also converges to the some point *z* in *X*.
Let us now suppose that *I* is continuous so that the sequence $\{I^2x_{2n}\}$ and $\{IABx_{2n}\}$ converge to the point *Iz*. Since *AB* and *I* are compatible of type (A), we have

$$\lim_{n\to\infty} d(ABIx_{2n}, IIx_{2n}) = 0$$

Hence it follows that the sequence $\{ABIx_{2n}\}$ also converges to *Iz*.
Now to show that $z = Iz$. we consider.

$$d(ABIx_{2n}, STx_{2n+1})$$

$$\leq \alpha\left[\frac{d(I^2x_{2n}, ABIx_{2n}).d(I^2x_{2n}, STx_{2n+1}) + d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABIx_{2n})}{d(I^2x_{2n}, ABIx_{2n}) + d(Jx_{2n+1}, STx_{2n+1})}\right]$$

$$+ \beta d(I^2x_{2n}, Jx_{2n+1})$$

which on letting $n \to \infty$, and in view of condition (2), we have $Iz = z$, Again we consider

$$d(ABz, STx_{2n+1})$$

$$\leq \alpha\left[\frac{d(Iz, ABz).d(Iz, STx_{2n+1}) + d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABz)}{d(Iz, ABz) + d(Jx_{2n+1}, STx_{2n+1})}\right]$$

$$+ \beta d(Iz, Jx_{2n+1})$$

and on letting $n \to \infty$, we get $ABz = z$

This means that *z* is in the range of *AB* and since $AB(X) \subset J(X)$ there exist a point z' in *X* such that $Jz' = z$. So that $STz = ST(Jz')$.
Now we consider

$$d(z, STz') = d(ABz, STz')$$

$$\leq \alpha\left[\frac{d(Iz, ABz).d(Iz, STz') + d(Jz', STz').d(Jz', ABz)}{d(Iz, ABz) + d(Jz', STz')}\right] + \beta d(Iz, Jz')$$

As $\beta < 1$, therefore we have $z = STz'$

Thus we have shown that $z = Jz' = STz'$ and since (*ST, J*) are compatible of type (A), one has $STz = ST(Jz') = J(Jz') = Jz$, which shows that

$\lim_{n\to\infty} d(STJz', JJz') = 0$,

which implies that
STz = *Jz*. Further

$$d(z, STz) = d(ABz, STz)$$
$$\leq \alpha\left[\frac{d(Iz, ABz).d(Iz, STz) + d(Jz, STz).d(Jz, ABz)}{d(Iz, ABz) + d(Jz, STz)}\right] + \beta d(Iz, Jz)$$

As $\beta < 1$, and in view of condition (2), we have *z* = *STz*.
Thus we have $z = STz = Jz$
Thus we have proved that $z = Iz = ABz = STz = Jz$ and so *z* is a common fixed point of *AB, I, ST* and *J*.
Now suppose that *AB* is continuous so that the sequence $\{AB^2x_{2n}\}$ and $\{ABIx_{2n}\}$ converges to *ABz* and since *AB* and *I* are compatible of type (A), we have

$\lim_{n\to\infty} d(IABx_{2n}, (AB)^2 x_{2n}) = 0$

which implies that the sequence $\{IABx_{2n}\}$ also converges to *ABz*. Again,

$$d(AB^2x_{2n}, STx_{2n+1})$$
$$\leq \alpha\left[\frac{d(IABx_{2n}, (AB)^2x_{2n}).d(IABx_{2n}, STx_{2n+1}) + d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, (AB)^2x_{2n})}{d(IABx_{2n}, (AB)^2x_{2n}) + d(Jx_{2n+1}, STx_{2n+1})}\right]$$
$$+ \beta d(IABx_{2n}, Jx_{2n+1})$$

and on letting $n \to \infty$, and using the condition (2), we get *ABz* = *z*
As earlier there exists z' in *X*, $AB(X) \subset J(X)$, such that $ABz = z = Jz'$, then

$$d(AB^2x_{2n}, STz')$$
$$\leq \alpha\left[\frac{d(IABx_{2n}, (AB)^2x_{2n}).d(IABx_{2n}, STz') + d(Jz', STz').d(Jz', (AB)^2x_{2n})}{d(IABx_{2n}, (AB)^2x_{2n}) + d(Jz', STz')}\right]$$
$$+ \beta d(IABx_{2n}, Jz')$$

which on letting $n \to \infty$, reduces to

$d(z, STz') = 0$

This gives $STz' = z = Jz'$. Since *ST* and *J* are compatible of type (A), then we have

$\lim_{n\to\infty} d(STJz', JJz') = 0$

giving there by *STz* = *Jz*. Further we consider

$$d(ABx_{2n}, STz)$$
$$\leq \alpha\left[\frac{d(Ix_{2n}, (AB)x_{2n}).d(Ix_{2n}, STz) + d(Jz, STz).d(Jz, (AB)x_{2n})}{d(Ix_{2n}, (AB)x_{2n}) + d(Jz, STz)}\right]$$
$$+ \beta d(Ix_{2n}, Jz)$$

Letting $n \to \infty$ and using the condition (2), we get *d*(*z, STz*) = 0 so it follows that $z = STz = Jz = ABz$

The point *z* therefore is in the range of *ST* and since $ST(X) \subset I(X)$, there exists a point z'' in *X* such that $Iz'' = z$. Thus

$$d(ABz'', z) = (ABz'', STz)$$
$$\leq \alpha\left[\frac{d(Iz'', ABz'')d(Iz'', STz) + d(Jz, STz).d(Jz, ABz'')}{d(Iz'', ABz'') + d(Jz, STz)}\right] + \beta d(Iz'', Jz)$$

and so $ABz'' = z$. Thus we have shown that $z = Iz'' = ABz''$ and from the compatibility of type (A) of *AB* and *I* it follows that

$\lim_{n\to\infty} d(ABIz'', IIz'') = 0$

giving thereby $ABz = Iz$
Thus, once again we have proved that
$z = ABz = Iz = STz = Jz$ and so z is common fixed point of AB, ST, I and J.
If the mappings ST or J is continuous instead of AB or I then the proof that z is a common fixed point of AB, ST, I and J is similar.
Let υ be another fixed point of I, J, AB and ST then,
$d(z, \upsilon) = d(ABz, ST\upsilon)$

$$\leq \alpha\left[\frac{d(Iz, ABz).d(Iz, ST\upsilon) + d(J\upsilon, ST\upsilon).d(J\upsilon, ABz)}{d(Iz, ABz) + d(J\upsilon, ST\upsilon)}\right] + \beta d(Iz, J\upsilon)$$

As $\beta < 1$ and in view of condition (2), we have $z = \upsilon$.
Finally, we need to show that z is also a common fixed point of A,B,S,T,I and J. For this let z be the unique common fixed point of both the pairs (AB,I) and (ST, J) then

$$Az = A(ABz) = A(BAz) = AB(Az), \quad Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz),$$
$$Bz = B(Iz) = I(Bz),$$

which show that Az and Bz is a common fixed point of (AB, I) yielding thereby $Az = z = Bz = Iz = ABz$ in the view of uniqueness of common fixed point of the pair (AB, I).
Similarly using the commutativity of (S,T), (S,J) and (T,J) it can be shown that $Sz = z = Tz = Jz = STz$
Now we need to show that $Az = Sz$ $(Bz = Tz)$ also remains a common fixed point of both the pairs (AB,I) and (ST, J) for this

$$d(Az, Sz) = d(A(BAz), S(TSz))$$
$$= d(AB(Az), ST(Sz))$$
$$\leq \alpha\left[\frac{d(I(Az), AB(Az)).d(I(Az), ST(Sz)) + d(J(Sz), ST(Sz))d(J(Sz), AB(Az))}{d(I(Az), AB(Az)) + d(J(Sz), ST(Sz))}\right]$$
$$+ \beta d(I(Az), J(Sz))$$

As $\beta < 1$ and using the condition (2), we have
$$d(Az, Sz) = 0$$
yielding thereby $Az = Sz$. Similarly it can be shown that $Bz = Tz$.
Thus z is the unique common fixed point of A,B,S,T,I, and J

Case – 2 :

$$U_{2n} + U_{2n+1} = 0 \ \ from \ \ some \ n, then$$

if $U_{2n} = d(ABx_{2n}, STx_{2n+1}) = 0$

$$and \ U_{2n+1} = d(STx_{2n+1}, ABx_{2n+2}) = 0$$

giving thereby

$$ABx_{2n} = Jx_{2n+1} = STx_{2n+1} = Ix_{2n+2} = ABx_{2n+2} = z$$

we assert that there exists a point w such that $ABw = Iw = STw = z$ otherwise if $ABw = Iw \neq z$.
Then

$$0 < d(Iw, z) = d(ABw, STx_{2n+1})$$
$$\leq \alpha\left[\frac{d(Iw, ABw).d(Iw, STx_{2n+1}) + d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABw)}{d(Iw, ABw) + d(Jx_{2n+1}, STx_{2n+1})}\right]$$
$$+ \beta d(Iw, Jx_{2n+1})$$

As $\beta < 1$, and in view of condition (2), we have $Iw = z$,
which yields thereby $Iw=ABw=z$. similarly, we shows that $STw=Jw=z$.
That z is a unique common fixed point of AB and I, ST and J.
The rest of the proof is identical to the case (1), hence it is omitted. This completes the proof.
By putting AB = S, ST = T, in our Theorem 1, we obtain the following result.

2.3 Corallary :

Let *S,T, I* and *J* be self mappings of a complete metric space (*X,d*) satisfying $S(X) \subset J(X), T(X) \subset I(X),$ and for each $x, y \in X$ $\alpha, \beta \geq 0, \alpha + \beta \leq 1$

either

$$d(Sx,Ty) \leq \alpha \left[\frac{d(Ix,Sx).d(Ix,Ty) + d(Jy,Ty).d(Jy,Sx)}{d(Ix,Sx) + d(Jy,Ty)} \right] + \beta d(Ix,Jy)$$

whenever $d(Ix,Sx) + d(Jy,Ty) \neq 0$ *or*

$d(Sx,Ty) = 0$ whenever $d(Ix,Sx) + d(Jy,Ty) = 0$

(*a*) (*S, I*) and (*T,J*) are compatible of type (*A*), if one of *S, T, I* and *J* is continuous then *S, T, I* and *J*, have a common fixed point *z* in *X* further z is the unique common fixed point of *S* and *I* and of *T* and *J*.

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