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Some Common Fixed Point Theorems For Asymptotically Commuting Mappings In Banach Spaces

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Abstract
Motivated and inspired by the result of Imdad, Ahamad and Khan [4] and contractive condition studied by Nesic [8], we have proved some common fixed point theorems for asymptotically commuting mappings in uniformly convex Banach spaces. Our work generalizes some known results with respect to their mappings and inequality conditions.

Keywords: *Fixed points, complete metric space, Banach spaces, asymptotically commuting, weakly commuting.*

2000 SUBJECT CLASSIFICATION CODE : 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES :

Let R_+ be the set of all non-negative reals and H_i be the family of all functions from R_+^i to R_+ for each positive integer *i*, which are upper semi continuous and non decreasing in each coordinate variable.

Now, the following definitions are borrowed by several authors the weak-commutativity condition introduced by Sessa [9] in metric space, which can be described in normed linear space stated as

1.1 Definition :
Let *A* and *S* be two self mappings of a normed linear space *X*. Then (*A, S*) is said to by **weakly commuting** pair on *X* if $\|SAx - ASx\| \le \|Ax - Sx\| \; for\, all \;\; x \in X$
obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

1.2 Example :

Let $X = [0,1]$ be the reals with Euclidean norm $Ax = \dfrac{x}{4+x}$ and $Sx = \dfrac{x}{2+x}$ for any $x \in X$.

$$\|SAx - ASx\| = \frac{x}{8+3x} - \frac{x}{8+5x} = \frac{2x^2}{(8+3x)(8+5x)}$$
$$\le \frac{2x}{(2+x)(4+x)} = \|Sx - Ax\|$$

So the pair (*A, S*) is weakly commuting but it is not commuting $SAx \ne ASx$.
The definition of compatible maps was given by Jungck [7], which can be stated as

1.3 Definition :
Let *A* and *S* be two self mappings of normed linear space *X*. Then (*A, S*) is said to be **asymptotically or preorbitally commuting** (also called compatible (Jungck [12]) its.

$$\lim_n \|ASx_n - SAx_n\| = 0 \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that}$$
$$\lim_n Ax_n = \lim_n Sx_n = u \text{ for some } u \text{ in } X.$$

The following example also supports the observation**1.4 Example :**
$Let\, X = [0,\infty)$, $Ax = 2x^2$, $5x = 3x^2$ and *d* the absolute value metric on *X* then *A* and *S* are not weakly commuting. However, for $x_n = 2^{-n}, d(Ax_n, Sx_n) \to 0, as \;\; n \to \infty$ and

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also
$d(ASx_n, SAx_n) \to 0, as \ \ n \to \infty$
evidently a weakly commuting pair is always asymptotically commuting but the converse is not true in general.
In (1974) Iscki [5] stated as

1.5 Definition :
The **modulus of convexity** of a Banach space *E* is a function $\delta : (0,2] \to (0,1]$ defined by

$$\delta(\in) = in \ f\left\{1 - \frac{1}{2}\|x + y\| : x, y \in E; \|x\| = \|y\| = 1, \|x - y\| \geq, \in\right\}$$

It is well known (Iscki [5]) that if E is uniformly convex then δ is strictly increasing, $Lim_{\in \to 0}\delta(\in) = 0$ and $\delta(2) = 1$. Let η denotes the inverse of δ, then we note that $\eta(t) < 2 \ \ for \ t < 1$.
We shall need the following Lemma of Goebel et al. [2].

1.6 Lemma :
Let *E* be a uniformly convex Banach space and B_γ the closed ball in *E* centered at origin with radius $\gamma > 0$, if $x_1, x_2, x_3 \in B_\gamma$

$$\|x_1 - x_2\| \geq \|x_2 - x_3\| \geq d > 0 \ and \ \|x_2\| \geq \left[1 - \frac{1}{2}\delta\left(\frac{d}{\gamma}\right)\right]\gamma$$

$$\text{then } \|x_1 - x_3\| \leq \eta\left[1 - \frac{1}{2}\delta\left(\frac{d}{\gamma}\right)\right]\|x_1 - x_2\|$$

2 MAIN RESULTS
Let R^+ be the set of non-negative real numbers, and let $F : R^+ \to R^+$ be mapping such that *F* (0) = 0 and *F* is continuous at 0.

2.1 Theorem :
Let *E* be a uniformly convex Banach space and *K*, a non-empty closed subset of *E*. Let {*S*,*I*} and {*T*,*J*} be two asymptotically commuting pairs of self-mappings of *K* such that for all $x, y \in K$.

$$\begin{aligned}\|Sx - Ty\|^2 \leq \phi \ (&\|Ix - Jy\| \, \|Ix - Sx\|, \|Ix - Jy\| \, \|Jy - Ty\|, \|Ix - Sx\| \\ &\|Jy - Ty\|, \|Ix - Ty\| \|Ix - Sx\|, \|Jy - Sx\| \|Jy - Ty\|) \\ &+ F\left(\min\{\|Jy - Sx\|, \|Jy - Ty\|, \|Ix - Sx\|, \|Ix - Ty\|\}\right)\end{aligned} \quad \text{....(1)}$$

where $\phi \in H_5$ and for all $t > 0$,
(i) $\phi(t,t,t,\alpha t,0) \leq \beta \ t$, and $\phi(t,t,t,0,\alpha t) \leq \beta \ t$

where β = 1 for α = 2 and β < 1 for α< 2 ;
(ii) ϕ (0,0,0,0,0) = 0;
(iii) *I* and *J* are continuous *S*(*K*) $\subset$ *J* (*K*) and *T*(*K*) $\subset$ *I*(*K*) ,

Then (a) *S*,*T*,*I* and *J* have a unique common fixed point *z* in *K* and
(b) for any $x_0 \in K$ the sequence generated by
$Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}, n = 0,1,2,........$
converges strongly to *z*.

Proof : Choose an arbitrary point x_0 in *K*. *As*, *S*(*K*) $\subset$ *J*(*K*), we can choose a point x_1 in *K* such that $Sx_0 = Jx_1$. Also since *T*(*K*) $\subset$ *I* (*K*), choose a another point x_2 in *K* such that $Tx_1 = Ix_2$. In this way, choose x_{2n}, x_{2n+1}, x_{2n+2} such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for *n* = 0,1,2,…

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Thus we get the sequence

$$\{Sx_0, Tx_1, Sx_2 .....Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}......\} \quad \ldots (2)$$

Let $d_{2n} = \|Sx_{2n} - Tx_{2n+1}\|$ and

$d_{2n+1} = \|Tx_{2n+1} - Sx_{2n+2}\|$, then

using inequality (i)
we have,

$$\begin{aligned}
&\|Sx_{2n} - Tx_{2n+1}\|^2 \\
&\le \phi\ (\|Ix_{2n} - Jx_{2n+1}\|\ \|Ix_{2n} - Sx_{2n}\|, \|Ix_{2n} - Jx_{2n+1}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|, \\
&\quad \|Ix_{2n} - Sx_{2n}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|, \|Ix_{2n} - Tx_{2n+1}\|\ \|Ix_{2n} - Sx_{2n}\|, \\
&\quad \|Jx_{2n+1} - Sx_{2n}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|) \\
&+ F\left(\min\{\|Jx_{2n+1} - Sx_{2n}\|, \|Jx_{2n+1} - Tx_{2n+1}\|, \|Ix_{2n} - Tx_{2n+1}\|, \|Ix_{2n} - Sx_{2n}\|\}\right)
\end{aligned}$$

which implies

$$\begin{aligned}
d_{2n}^2 &\le \phi\left(d_{2n-1}^2, d_{2n-1}d_{2n}, d_{2n-1}d_{2n}, (d_{2n-1} + d_{2n})d_{2n-1}, 0\right) \\
&\quad + F\left(\min\{0, d_{2n}, (d_{2n-1} + d_{2n}), d_{2n-1}\}\right)
\end{aligned}$$

or $$d_{2n}^2 \le \phi\left(d_{2n-1}^2, d_{2n-1}d_{2n}, d_{2n-1}d_{2n}, \alpha\, d_{2n-1}, 0\right) + F(0)$$

or $$d_{2n}^2 \le \phi\left(d_{2n-1}^2, d_{2n-1}d_{2n}, d_{2n-1}d_{2n}, \alpha\, d_{2n-1}, 0\right) \quad \ldots.(3)$$

Similarly, we obtain

$$\begin{aligned}
&\|Sx_{2n+2} - Tx_{2n+1}\|^2 \\
&\le \phi\ (\|Ix_{2n+2} - Jx_{2n+1}\|\ \|Ix_{2n+2} - Sx_{2n+2}\|, \|Ix_{2n+2} - Jx_{2n+1}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|, \\
&\quad \|Ix_{2n+2} - Sx_{2n+2}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|, \|Ix_{2n+2} - Tx_{2n+1}\|\ \|Ix_{2n+2} - Sx_{2n+2}\|, \\
&\quad \|Jx_{2n+1} - Sx_{2n+2}\|\ \|Jx_{2n+1} - Tx_{2n+1}\|) \\
&+ F\left(\min\{\|Jx_{2n+1} - Sx_{2n+2}\|, \|Jx_{2n+1} - Tx_{2n+1}\|, \|Ix_{2n+2} - Tx_{2n+1}\|, \|Ix_{2n+2} - Sx_{2n+2}\|\}\right)
\end{aligned}$$

$$\begin{aligned}
d_{2n+1}^2 &\le \phi\left(d_{2n}d_{2n+1}, d_{2n}^2, d_{2n}d_{2n+1}, 0, (d_{2n} + d_{2n+1})d_{2n}\right) \\
&\quad + F\left(\min\{(d_{2n} + d_{2n+1}), d_{2n}, 0, d_{2n+1}\}\right)
\end{aligned}$$

or $$d_{2n+1}^2 \le \phi\left(d_{2n}d_{2n+1}, d_{2n}^2, d_{2n}d_{2n+1}, 0, \alpha' d_{2n}\right) + F(0)$$

or $$d_{2n+1}^2 \le \phi\left(d_{2n}d_{2n+1}, d_{2n}^2, d_{2n}d_{2n+1}, 0, \alpha' d_{2n}\right) \quad \ldots (4)$$

Suppose for some $n\, d_{2n+1} > d_{2n} > d_{2n-1}$, then $d_{2n-1} + d_{2n} = \alpha d_{2n}$ with some $1 < \alpha < 2$ and $d_{2n} + d_{2n+1} = \alpha^1 d_{2n+1}$ with some $1 < \alpha^1 < 2$ since in each coordinate, variable ϕ is non decreasing.

$$\begin{cases} d_{2n}^2 \le \phi\left(d_{2n}^2, d_{2n}^2, d_{2n}^2, \alpha d_{2n}^2, 0\right) \\ d_{2n+1}^2 \le \phi\left(d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2, 0, \alpha d_{2n+1}^2\right) \end{cases} \quad \ldots (5)$$

In both the cases by (i), we have

$$d_{2n}^2 \le \beta d_{2n}^2, \frac{1}{2} < \beta < 1,$$

$$d_{2n+1}^2 \le \beta' d_{2n+1}^2, \frac{1}{2} < \beta' < 1,$$

which is contradiction. Therefore,

$d_{2n} \le d_{2n-1}$ *i.e.,* $d_{2n} \ge d_{2n+1}, \quad n = 1,2....$

suppose further that $\lim_n \{d_{2n}\} = \lim_n \{d_{2n+1}\} = d \ge 0,$

we claim that d=0 and if not we can say d> 0, without loss of generality.

We can postulate that $0 \in K$ and $0 < \gamma'$ = sup $\|d_{2n}\|$

Let, $\gamma \in R_+$ be chosen in such a way that

$\gamma' < \gamma$ *and* $\gamma\left[1 - \frac{1}{2}\delta\left(\frac{d}{\gamma}\right)\right] < \gamma'$,

we can find a sequence $\{\eta_i\}$, i = 0,1,2……. of positive integers such that for i.e. j $\in \{\eta_i\}$

$d_{2j} \ge \gamma\left[1 - \frac{1}{2}\delta\left(\frac{d}{\gamma}\right)\right]$ while for $\eta > \eta_0, \quad d_{2\eta} \le \gamma$

since $d_{2\eta_j - 1} \ge d_{2\eta_j} \ge d \ge 0$ for every $i = 0,1,2,....$

It follows from Lemma 1.6, it follows that for any $j \in \{\eta_i\}$

$$\|Sx_{2j-2} - Sx_{2j}\| \le \|Sx_{2j-2} - Tx_{2j-1}\| + \|Tx_{2j-1} - Sx_{2j}\|$$

$$\le \eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{\gamma}\right)\right)\|Sx_{2j-2} - Tx_{2j-1}\|$$

$$\le \eta\left(\frac{\gamma'}{\gamma}\right)d_{2j-1} = \alpha_1 d_{2j} - 1 \qquad \text{… (6)}$$

where $\alpha_1 = \eta\left(\frac{\gamma'}{\gamma}\right) < 2$ because of uniform convexity.

Then we have by (3), (4), (5) and (6)

$$\begin{Bmatrix} d_{2j}^2 \le \phi\left(d_{2j-1}^2, d_{2j-1}^2, d_{2j-1}^2, \alpha_1 d_{2j-1}^2, 0\right) \\ d_{2j+1}^2 \le \phi\left(d_{2j}^2, d_{2j}^2, d_{2j}^2, 0, \alpha_1 d_{2j}^2\right) \end{Bmatrix} \qquad \text{… (7)}$$

Thus in either case $d_{2j} \le \beta_1 d_{2j-1}$ and $d_{2j+1} \le \beta_1 d_{2j}$ for some $\beta_1 < 1$.

We observed that β_1 is independent of j and so, as $j \to \infty$, we have $d \le \beta_1 d$, a evident contradiction implying at d = 0.

It follows therefore, as proved in (Husain and Sehgal [3] that the sequence (2) is cauchy sequence. But K is closed subset of E, therefore sequence (2) converges to a point z in K, hence the sequence $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ which are subsequences of (1) also converges to the point z.

Since I is continuous then sequence I^2x_{2n} and ISx_{2n} converges to Iz. since $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ix_{2n} = z$ and (S,I) is asymptotically commuting, then

$$Lim_{n\to\infty}\|SIx_{2n} - ISx_{2n}\| = 0$$

which implies that $SIx_{2n} \to Iz$

Taking $x = Ix_{2n}, \quad y = x_{2n+1}$, in condition (1)

$$\begin{aligned}
&\|SIx_{2n}-Tx_{2n+1}\|^2\\
&\le \phi\big(\|I^2x_{2n}-Jx_{2n+1}\|\,\|I^2x_{2n}-SIx_{2n}\|,\|I^2x_{2n}-Jx_{2n+1}\|\,\|Jx_{2n+1}-Tx_{2n+1}\|,\\
&\quad \|I^2x_{2n}-SIx_{2n}\|\|Jx_{2n+1}-Tx_{2n+1}\|,\|I^2x_{2n}-Tx_{2n+1}\|\,\|I^2x_{2n}-SIx_{2n}\|,\\
&\quad \|Jx_{2n+1}-SIx_{2n}\|\,\|Jx_{2n+1}-Tx_{2n+1}\|\big)+F\big(\min\{\|Jx_{2n+1}-SIx_{2n+1}\|,\\
&\quad \|Jx_{2n+1}-Tx_{2n+1}\|,\|I^2x_{2n}-SIx_{2n}\|,\|I^2x_{2n}-Tx_{2n+1}\|\}\big)
\end{aligned}$$

Taking $\lim n\to\infty$, we have

$$\begin{aligned}
\|Iz-z\|^2 &\le \phi\big(\|Iz-z\|\,\|Iz-Iz\|,\|Iz-z\|\,\|z-z\|,\|Iz-Iz\|\,\|z-z\|,\\
&\qquad \|Iz-z\|\,\|Iz-Iz\|,\ \|z-Iz\|\,\|z-z\|\ \big)\\
&\qquad +F\big(\min\{\|z-Iz\|,\|z-z\|,\|Iz-Iz\|,\|Iz-z\|\ \}\big)
\end{aligned}$$

or $$\|Iz-z\|^2 \le \phi(0,0,0,0,0)+F\big(\min\{\|z-Iz\|,0,0,\|Iz-z\|\ \}\big)$$

or $$\|Iz-z\|^2 \le 0+F(0)$$

which implies that $Iz=z$.

Since *J* in continuous and (*T*,*J*) in asymptotically commuting. So the sequence $J^2x_{2n+1}\to Jz$, $JTx_{2n+1}\to Jz$.
Since $\lim_{n\to\infty}Tx_{2n+1}=\lim_{n\to\infty}Jx_{2n+1}=z$. while (*T*,*J*) is asymptotically commuting then
$\lim_{n\to\infty}\|TJx_{2n+1}-JTx_{2n+1}\|=0$,
which implies that $TJx_{2n+1}\to Jz$
Now, putting $x=x_{2n}$, $\quad y=Jx_{2n+1}$, in condition (1), we have

$$\begin{aligned}
&\|Sx_{2n}-TJx_{2n+1}\|^2\\
&\le \phi\ \big(\|Ix_{2n}-J^2x_{2n+1}\|\,\|Ix_{2n}-Sx_{2n}\|,\|Ix_{2n}-J^2x_{2n+1}\|\,\|J^2x_{2n+1}-TJx_{2n+1}\|,\\
&\quad \|Ix_{2n}-Sx_{2n}\|\|J^2x_{2n+1}-TJx_{2n+1}\|,\|Ix_{2n}-TJx_{2n+1}\|\,\|Ix_{2n}-Sx_{2n}\|,\\
&\quad \|J^2x_{2n+1}-Sx_{2n}\|\,\|J^2x_{2n+1}-TJx_{2n+1}\|\big)\\
&+F\big(\min\{\|JTx_{2n+1}-Sx_{2n}\|,\ \|J^2x_{2n+1}-TJx_{2n+1}\|,\|Ix_{2n}-Sx_{2n}\|,\|Ix_{2n}-TJx_{2n+1}\|\ \}\big)
\end{aligned}$$

$$\begin{aligned}
\|z-Jz\|^2 &\le \phi\big(\|z-Jz\|\,\|z-z\|,\|z-Jz\|\,\|Jz-Jz\|,\|z-z\|\,\|Jz-Jz\|,\\
&\qquad \|z-Jz\|\,\|z-z\|,\|Jz-z\|\,\|Jz-Jz\|\ \big)\\
&\qquad +F\big(\min\{\|Jz-z\|,\|Jz-Jz\|,\|z-z\|,\|z-Jz\|\}\big)
\end{aligned}$$

or $$\|z-Jz\|^2 \le \phi(0,0,0,0,0)+F\big(\min\{\|Jz-z\|,0,0,\|z-Jz\|\}\big)$$

or $$\|z-Jz\|^2 \le 0+F(0)$$

giving there by $z = Jz$ which implies $z = Jz = Iz$
Taking $x = z$, $y = x_{2n+1}$, in condition (1), we have

$$\begin{aligned}
&\|Sz-Tx_{2n+1}\|^2\\
&\le \phi\big(\|Iz-Jx_{2n+1}\|\,\|Iz-Sz\|,\ \|Iz-Jx_{2n+1}\|\ \ \|Jx_{2n+1}-Tx_{2n+1}\|,\|Iz-Sz\|\\
&\quad \|Jx_{2n+1}-Tx_{2n+1}\|,\|Iz-Tx_{2n+1}\|\,\|Iz-Sz\|,\|Jx_{2n+1}-Sz\|\,\|Jx_{2n+1}-Tx_{2n+1}\|\big)\\
&+F\big(\min\{\|Jx_{2n+1}-Sz\|,\ \|Jx_{2n+1}-Tx_{2n+1}\|,\|Iz-Sz\|,\ \|Iz-Tx_{2n+1}\|\ \}\big)
\end{aligned}$$

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Taking $\lim n \to \infty$, we have

$$\text{or} \quad \|Sz - z\|^2 \le \phi(0,0,0,0,0) + F\left(\min\left\{\|z - Sz\|, \|z - z\|, \|Iz - Sz\|, \|Iz - z\|\right\}\right)$$

$$\text{or} \quad \|Sz - z\|^2 \le 0 + F(0)$$

yielding thereby Sz = z.
Now, taking $x = x_{2n}$, $y = z$, in condition (1), we have

$$\begin{aligned}
&\|Sx_{2n} - Tz\|^2 \\
&\le \phi\big(\|Ix_{2n} - Jz\|\,\|Ix_{2n} - Sx_{2n}\|, \|Ix_{2n} - Jz\|\,\|Jz - Tz\|, \|Ix_{2n} - Sx_{2n}\| \\
&\quad \|Jz - Tz\|, \|Ix_{2n} - Tz\|\,\|Ix_{2n} - Sx_{2n}\|, \|Jz - Sx_{2n}\|\,\|Jz - Tz\|\big) \\
&+ F\left(\min\left\{\|Jz - Sx_{2n}\|, \|Jz - Tz\|, \|Ix_{2n} - Sx_{2n}\|, \|Ix_{2n} - Tz\|\right\}\right)
\end{aligned}$$

Taking $\lim n \to \infty$, we have

$$\begin{aligned}
&\|z - Tz\|^2 \\
&\le \phi\big(\|z - z\|\|z - z\|, \|z - z\|\,\|z - Tz\|, \|z - z\|\|z - Tz\|, \|z - Tz\|\,\|z - z\|, \\
&\|z - z\|\|z - Tz\|\big) + F\left(\min\left\{\|z - z\|, \|z - Tz\|, \|z - z\|\|z - Tz\|\right\}\right)
\end{aligned}$$

$$\text{or} \quad \|z - Tz\|^2 \le \phi(0,0,0,0,0) + F\left(\min\left\{0, \|z - Tz\|, 0, \|z - Tz\|\right\}\right)$$

$$\text{or} \quad \|z - Tz\|^2 \le 0 + F(0)$$

yielding thereby $z = Tz$ which implies $z = Sz = Tz$. Thus we have proved that $z = Sz = Iz = Tz = Jz$, So z is the common fixed point of S, I, T and J. This completes the proof.
If we take $F(t) = 0$ and for all $t \in R^+$, in Theorem 2.1, we obtain the following result.

2.2 Corallary :
Let E be uniformly convex Banach space and K a non empty closed subset of E. Let $\{S,I\}$ and $\{T, J\}$ be two asymptotically commuting pairs of self mappings of K such that for all $x, y \in K$.

$$\begin{aligned}
\|Sx - Ty\|^2 \le \phi(&\|Ix - Jy\|\,\|Ix - Sx\|, \|Ix - Jy\|\,\|Jy - Ty\|, \|Ix - Sx\| \\
&\|Jy - Ty\|, \|Ix - Ty\|\,\|Ix - Sx\|, \|Jy - Sx\|\,\|Jy - Ty\|)
\end{aligned}$$

where $\phi \in H_5$ and for all $t > 0$,

(i) $\phi(t,t,t,\alpha t,0) \le \beta t$ and $\phi(t,t,t,0,\alpha t) \le \beta t$ where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;

(ii) $\phi(0,0,0,0,0) = 0;$

(iii) I and J are continuous $S(K) \subset J(K)$ and $T(K) \subset I(K)$

Then (a) S, T, I and J have a unique common fixed point z in K and.

(b) for any $x_0 \in K$ the sequence generated by

$$Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}, \qquad n = 0,1,2...$$

converges strongly to z.
the above corallary (2.2) shows the result of Imdad, Ahmad and Khan [4].
The following theorem also generalizes the result of Som [10] and Imdad, Ahmad and Khan [4].

2.3 Theorem :
Let E be uniformly convex Banach space and K a non empty closed subset of E. Let S,T,I and J be four self mapping of K such that for all $x, y \in K$.

$$\|Sx-Ty\|^2 \le \phi\big(\|Ix-Sx\|\,\|Jy-Ty\|, \|Ix-Ty\|\,\|Jy-Sx\|, \|Ix-Sx\|\,\|Ix-Ty\|, \|Jy-Sx\|\,\|Jy-Ty\|\big) + F\left(\min\{\|Jy-Sx\|, \|Jy-Ty\|, \|Ix-Sx\|, \|Ix-Ty\|\}\right)$$

where $\phi \in H_4$ and for all $t > 0$,

(i) $\phi(t,0,\alpha t,0) \le \beta t$ and $\phi(t,0,0,\alpha t) \le \beta t$ where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;

(ii) $\phi(0,t,0,0) < t, \phi(0,0,0,0) = 0$;

(iii) $S(K) \subset J(K)$ and $T(K) \subset I(K)$;

(iv) *I* is continuous, (*S*,*I*) is asymptotically commuting and (*T*,*J*) is weakly commuting pair in *K* or *J* is continuous, (*T*,*J*) is asymptotically commuting and (*S*,*I*) is weakly commuting pair in *K*. Then

(a) *S*, *T*, *I* and *J* have a unique common fixed point *z* in *K* and.

(b) for any $x_0 \in K$ the sequence generated by

$$Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}, \quad n = 0,1,2...$$

converges strongly to *z*.

Proof : It may be completed on the lines of proof of Theorem 2.1

If we take I = J and F (t) = 0, for all $t \in R^+$ in Theorem 2.3, we get the following result of Som [10].

2.4 Corallary :

Let *E* be a uniformly convex Banach space and *K* a non-empty closed subset of *E*. Let *S*, *T* and *I* be three self mappings of *K* such that for all *x*, $y \in K$.

$$\|Sx-Ty\|^2 \le \phi\big(\|Ix-Sx\|\,\|Iy-Ty\|, \|Ix-Ty\|\,\|Iy-Sx\|, \|Ix-Sx\|\,\|Ix-Ty\|, \|Iy-Sx\|\,\|Iy-Ty\|\big)$$

where $\phi \in H_4$ and for all $t > 0$

(i) $\phi(t,0,\alpha t,0) \le \beta t$ and $\phi(t,0,0,\alpha t) \le \beta t$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;

(ii) $\phi(t,t,t,t) < t$;

(iii) *I* is continuous $S(K) \cup T(K) \subset I(K)$

(iv) {*S*, *I*} and {*T*, *I*} are asymptotically commuting pairs on *K*. Then there exists a point $u \in K$ such that

(a) *u* is the unique common fixed point of *S*,*T* and *I*;

(b) for any $x_0 \in K$, the sequence $\{Ix_n\}$ defined by

$Ix_{2n+1} = Sx_{2n}, Ix_{2n+2} = Tx_{2n+1}, \; n = 0,1,2...$ converges strongly to *u*.

Finally, we furnish the example to discuss the validity of foregoing Theorem 2.1

2.5 Example : Let *E* = *K* [0, 1] and define

$$S,T,I,J : K \to K \ as \ Sx = \frac{x^2}{3}, Tx = \frac{x^2}{4}, \quad Jx = \frac{x}{2} \ and \ Ix = \frac{3x}{4}$$

$$Note\ that\ S(K) \subset J(K)$$

$$\Rightarrow S(K) = \left[0, \frac{1}{3}\right] \subset \left[0, \frac{1}{2}\right] = J(K)$$

$$Let\ \phi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{5}(t_1+t_2+t_3+t_4+t_5)$$

Then above function satisfy condition (4.1) for all x, y in [0,1]. Clearly 0 is the unique fixed point of *S*,*T*,*I* and *J*. taking *F* (*t*) = 0 for all $t \in R^+$

Next example proof the validity of corallary (2.4)

2.6 Example :

Let K = [0,1]. Define the mappings *S, T* and *I* of *K* into itself by $Sx = \frac{x^2}{4}, Tx = \frac{x^2}{3}$ *and* $Ix = \frac{3x}{4}$,

Let $\phi(t_1, t_2, t_3, t_4) = \frac{6}{25}(t_1 + t_2 + t_3 + t_4)$ then all the conditions of corallary (2.4) are satisfied and 0 is the unique common fixed point of *S, T* and *I*.
Next example proof the validity of Theorem 2.3

2.7 Example :

Let K = [0,1]. Define the mappings *S, T, I* and *J* of K into itself by $Sx = \frac{x^2}{3}, Tx = \frac{x^2}{4}, \ Ix = \frac{3x}{4}$ *and* $Jx = \frac{x}{2}$

If we set $\phi(t_1, t_2, t_3, t_4) = \frac{1}{4}(t_1 + t_2 + t_3 + t_4)$ then all the conditions of Theorem 2.3 are satisfied for all $x, y \in [0,1]$. Clearly 0 is the unique common fixed point of *S, T , I* and *J*.

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