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On Arithmetical Functions

[1]Veena Narayanan, [2]R.Srikanth

Abstract--*In this paper, we provide some relations involving arithmetic functions φ, σ and ψ. The present work discusses some equalities regarding these arithmetic functions using the concept of maximum and minimum prime divisor.*

Key words--*Arithmetic functions, Euler totient function, Fundamental theorem of Arithmetic, Prime divisor, Prime factorization.*

I. INTRODUCTION

In [1, 5, and 6] arithmetic function is a function whose domain of definition is a set of positive integers. Numerous studies have been done on various properties of arithmetic functions worldwide. Some well-known integer valued arithmetic functions are Euler totient function (φ) which gives the number of integers less than *n*, and relatively prime to *n*and σ function which gives the sum of divisors of a positive integer. Let $n = {p_1}^{k_1}{p_2}^{k_2} \dots {p_i}^{k_i}$ where $p_1, p_2, \dots, p_i$, are primes and $k_1, k_2, \dots, k_i$, are positive, be the prime factorization of $n > 1$. Then we define

$$\varphi(n) = n\prod_{j=1}^{i}(1-\frac{1}{p_j})$$

$$\psi(n) = n\prod_{j=1}^{i}(1+\frac{1}{p_j})$$

and

$$\sigma(n) = \prod_{j=1}^{i}\frac{{p_j}^{k_j+1}-1}{p_j-1}$$

In 2006, Atanassov [1] developed some properties of arithmetic functions φ, σ and ψ using the concept of maximal prime divisor. But the author [1]did not explained the concept of maximal prime divisor and also the equalities proved in that article did not satisfy for the cases of $n = 6,10$ .... Later on 2010, Atanassov[2] introduced the idea of "pine tree" representation of these arithmetic functions. In 2011, again Atanassov[3] established some equality for these functions. In continuation of this work, Srikanth and Kannan [4] improved the bounds obtained in [3].

In the present article we are discussing some equalities regarding these arithmetic functions using the concept of maximum and minimum prime divisor. We also try to modify the idea of maximal prime divisor of the article [1].

[1] *Department of Mathematics, SASTRA Deemed to be University,Thanajvur, Tamil Nadu, India. Email:veenanarayanan@sastra.ac.in*
[2] *TATA Realty- SASTRA Srinivasa Ramanujan Research Chair for Mathematics,SASTRA Deemed to be University, Thanjavur, Tamil Nadu, India. Email: srikanth@maths.sastra.edu*

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Note that throughout this article q is a prime number.

II. RESULTS AND DISCUSSIONS

2.1. Theorem. Let $n > 1$ be a positive integer. Then we have

$$\max_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\sigma\left(\frac{n}{d}\right) = \varphi(\max_{q|n} q)\sigma\left(\frac{n}{\max_{q|n} q}\right)$$

Proof.

Let $n > 1$ be a positive integer. Then by the fundamental theorem of arithmetic, we can represent

$n = {p_1}^{k_1}{p_2}^{k_2} \dots {p_i}^{k_i}$ where $p_1, p_2, \dots, p_i$, are primes and $k_1, k_2, \dots, k_i$, are positive. Let *d*be any divisor of *n*. Then *d* takes the form $d = {p_1}^{l_1}{p_2}^{l_2} \dots {p_i}^{l_i}$ where $p_1, p_2, \dots, p_i$, are primes and $1 \leq l_i \leq k_i$. Then

$$\frac{n}{d} = {p_1}^{k_1-l_1}{p_2}^{k_2-l_2} \dots {p_i}^{k_i-l_i}$$

Therefore,

$$\varphi(d)\sigma\left(\frac{n}{d}\right) = \varphi({p_1}^{l_1})\sigma({p_1}^{k_1-l_1}) \dots \varphi({p_i}^{l_i})\sigma({p_i}^{k_i-l_i})$$

Without loss of generality we can take $p_1$be the maximum prime divisor of *n* and $l_1, l_2, \dots, l_i$ are positive. So,

(1) $$\varphi(d)\sigma\left(\frac{n}{d}\right) = {p_1}^{l_1-1}({p_1}^{k_1-l_1+1} - 1) \dots {p_i}^{l_i-1}({p_i}^{k_i-l_i+1} - 1)$$

and $$\varphi(\max_{q|n} q)\sigma\left(\frac{n}{\max_{q|n} q}\right) = \varphi(p_1)\sigma(\frac{n}{p_1})$$

Also we have

$$\varphi(p_1)\sigma\left(\frac{n}{p_1}\right) = ({p_1}^{k_1} - 1)\frac{{p_2}^{k_2+1}-1}{p_2-1} \dots \frac{{p_i}^{k_i+1}-1}{p_i-1}.$$

So by equation (1) it is clear that $\varphi(d)\sigma\left(\frac{n}{d}\right) \leq \varphi(p_1)\sigma\left(\frac{n}{p_1}\right)$ is true for all divisors*d*. This proves the result.

2.1.1. Example

Let $n = 10$. Then $n = 2 \cdot 5$. The maximum prime divisor of $n = 5$. Then,

$$\varphi(p_1)\sigma\left(\frac{n}{p_1}\right) = \varphi(5)\sigma(2) = 4 \cdot 3 = 12$$

Also from the Table 1, we get $\max_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\sigma\left(\frac{n}{d}\right) = 1$

Table 1: Illustration for Theorem 2.1

D	1	2	5	10
φ(d)	1	1	4	4
σ(n/d)	18	6	3	1

2.1.2. Note Theorem 2.1 does not hold if we consider $1 \leq d \leq n$. The above example (Example 2.1.1) also interprets this.

Theorem 2.1 also holds for φ and ψ. That is

2.2. Theorem. For every positive integer $n > 1$, we have

$$\max_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\psi\left(\frac{n}{d}\right) = \varphi(\max_{q|n} q)\psi\left(\frac{n}{\max_{q|n} q}\right)$$

Proof.

The proof is exactly similar to that of Theorem 1.

As in the case of Theorem 1, Theorem 2 also not valid for *d=1*. The following example supports this.

2.2.1. Example

Let $n = 10$. Then $n = 2 \cdot 5$. The maximum prime divisor of $n = 5$. Then,

$$\varphi(p_1)\psi\left(\frac{n}{p_1}\right) = \varphi(5)\psi(2) = 4 \cdot 3 = 12$$

Also from the Table 2, we get $\max_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\psi\left(\frac{n}{d}\right) = 12.$

Table 2: Illustration for Theorem 2.2

D	1	2	5	10
φ(d)	1	1	4	4
ψ(n/d)	18	6	3	1

2.3. Remark. Theorem 2.1 and Theorem 2.2 does not hold for ψ and σ functions. The following example illustrates this.

2.3.1. Example

Let $n = 36$. Then $n = 2^2 \cdot 3^2$. The maximum prime divisor of $n = 3$. Then,

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$$\psi(p_1)\sigma\left(\frac{n}{p_1}\right) = \psi(3)\sigma(12) = 4 \cdot 28 = 112$$

Also from the Table 3, we get $\max\limits_{\substack{d|n \\ 1<d\leq n}} \psi(d)\sigma\left(\frac{n}{d}\right) = 117$

Table 3: Illustration for Remark 2.3

d	1	2	3	4	9	12	18	36
ψ(d)	1	3	4	6	12	24	36	72
σ(n/d)	91	39	28	13	7	4	3	1

Instead of maximum prime divisor, if we consider minimum prime divisor we get inequalities rather than equalities. So if we use the idea of minimum prime divisor, then the above theorems becomes

2.4. Theorem. For every positive integer $n > 1$, we have

$$\min_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\sigma\left(\frac{n}{d}\right) \leq \varphi(\min_{q|n} q)\sigma\left(\frac{n}{\min\limits_{q|n} q}\right)$$

Proof.

Let $n > 1$ be a positive integer. Then by the fundamental theorem of arithmetic, we can represent

$n = {p_1}^{k_1}{p_2}^{k_2} \dots {p_i}^{k_i}$ where $p_1 \leq p_2 \leq \cdots \leq p_i$, are the prime factors and$k_1, k_2, \dots, k_i$, are positive. Let *d*be any divisor of *n*. Then *d* takes the form $d = {p_1}^{l_1}{p_2}^{l_2} \dots {p_i}^{l_i}$ where $p_1, p_2, \dots, p_i$, are primes and $1 \leq l_i \leq k_i$. Then

$$\frac{n}{d} = {p_1}^{k_1-l_1}{p_2}^{k_2-l_2} \dots {p_i}^{k_i-l_i}$$

Therefore,

$$\varphi(d)\sigma\left(\frac{n}{d}\right) = \varphi({p_1}^{l_1})\sigma({p_1}^{k_1-l_1}) \dots \varphi({p_i}^{l_i})\sigma({p_i}^{k_i-l_i})$$

Here p_1 is the minimum prime divisor of *n*and $l_1, l_2, \dots, l_i$ are positive. So,

(1) $$\varphi(d)\sigma\left(\frac{n}{d}\right) = {p_1}^{l_1-1}({p_1}^{k_1-l_1+1} - 1) \dots {p_1}^{l_i-1}({p_1}^{k_i-l_i+1} - 1)$$

and $$\varphi(\min_{q|n} q)\sigma\left(\frac{n}{\min\limits_{q|n} q}\right) = \varphi(p_1)\sigma(\frac{n}{p_1})$$

Also we have

$$\varphi(p_1)\sigma\left(\frac{n}{p_1}\right) = ({p_1}^{k_1} - 1)\frac{{p_2}^{k_2+1}-1}{p_2-1} \dots \frac{{p_i}^{k_i+1}-1}{p_i-1}.$$

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So by equation (1) it is clear that $\varphi(d)\sigma\left(\frac{n}{d}\right) \leq \varphi(p_1)\sigma\left(\frac{n}{p_1}\right)$ is true for all divisors d. Thus

$$\min_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\sigma\left(\frac{n}{d}\right) \leq \varphi(p_1)\sigma\left(\frac{n}{p_1}\right)$$

This proves the result.

2.4.1. Example

Let $n = 6$. Then the prime factorization of $n$yields $n = 2 \cdot 3$. In this case the minimum prime divisor is 2.

$$\varphi(\min_{q|n} q)\sigma\left(\frac{n}{\min_{q|n} q}\right) = \varphi(2)\sigma(3) = 1 \cdot 4 = 4$$

Also from Table 4, we get $\min_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\sigma\left(\frac{n}{d}\right) =$

Table 4: Illustration for Theorem 2.4

d	1	2	3	6
φ(d)	1	1	2	2
σ(n/d)	12	3	4	1

2.5. Theorem. For every positive integer $n > 1$, we have

$$\min_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\psi\left(\frac{n}{d}\right) \leq \varphi(\min_{q|n} q)\psi\left(\frac{n}{\min_{q|n} q}\right)$$

Proof.

The proof is exactly similar to that of Theorem 1.

2.5.1. Example

Let $n = 10$. Then $n = 2 \cdot 5$. The minimum prime divisor of $n = 2$. Then,

$$\varphi(p_1)\psi\left(\frac{n}{p_1}\right) = \varphi(2)\psi(5) = 1 \cdot 6 = 6$$

Also from the Table 2, we get $\min_{\substack{d|n \\ 1<d\leq n}} \varphi(d)\psi\left(\frac{n}{d}\right) = 4.$

2.6. Remark. In the case of minimum prime divisor also, equality as in Theorem 1 does not hold for ψ and σ. For example, consider the case $n = 36$. Then the minimum prime divisor is 2 and

$$\psi(p_1)\sigma\left(\frac{n}{p_1}\right) = 3 \cdot 39 = 117$$

From Table 3, we obtain $\min_{\substack{d|n \\ 1<d\leq n}} \psi(d)\sigma\left(\frac{n}{d}\right) = 72.$

III. CONCLUSION

In this paper, some extremal properties of arithmetic functionsφ, σ and ψis presented by considering both maximum prime divisor and minimum prime divisor. But no extremal properties are found out for the case of ψ and σ. So it is possible to extend this study for obtaining extremal properties for ψ and σ.

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