A STUDY ON SOME CLASSES OF TOPOGENIC GRAPH

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ABSTRACT-- *We give the definition of topogenic graph, some classes of topogenic graph and also we propose two open problems.*

KEYWORDS— *a study on some classes of topogenic graph*

Definition 1

We identify the initial vertex of P_n and the central vertex of $K_{1,m}$. The resulting graph is denoted by $P_n \odot K_{1,m}$.

Theorem 2

$P_3 \odot K_{1,m}$ is topogenic for every positive integer m.

Proof

Let $V(K_{1,m}) = \{u_1, v_1, v_2, \dots, v_m\}$ and
$V(P_3) = \{u_1, u_2, u_3\}$.
Then $V(P_3 \odot K_{1,m}) = V(P_3) \cup V(K_{1,m})$.
Choose $X = \{1,2, \dots, m+2\}$ as the ground set.
Let $f: V(P_3 \odot K_{1,m}) \to 2^X$ be defined by

$$f(v_i) = \{1,2,..i+2\}, 1 \le i \le m,$$

$f(u_1) = \emptyset, f(u_2) = \{1\}, f(u_3) = \{1,2\}$.
Then, both f and $f^{\oplus}$ are injective.
Therefore, $f(V) \cup f^{\oplus}(E)$ forms a topology on X.
Hence $P_3 \odot K_{1,m}$ is topogenic for every positive integer m.

Theorem 3

$P_4 \odot K_{1,m}$ is topogenic for every positive integer m.

Proof

Let $V(K_{1,m}) = \{u_1, v_1, v_2, \dots, v_m\}$ and
$V(P_4) = \{u_1, u_2, u_3, u_4\}$.
Then $V(P_4 \odot K_{1,m}) = V(P_4) \cup V(K_{1,m})$.
Without loss of generality, choose $X = \{1,2, \dots, m+3\}$ as the ground set.
Let $f: V(P_4 \odot K_{1,m}) \to 2^X$ be defined by

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$$f(v_i) = \{1,2,..i+3\}, 1 \leq i \leq m,$$

$f(u_1) = \emptyset, f(u_2) = \{1\}, f(u_3) = \{2\}, f(u_4) = \{1,2,3\}$.

Then, both f and $f^{\oplus}$ are injective.

Moreover, $f(v_1) \subset f(v_2) \subset \cdots \subset f(v_m)$ and

$f(u_2) \subset f(u_4)$ and $f(u_3) \subset f(u_4)$;

$f(v_i) \cap f(u_1) = \emptyset \in f(V), 1 \leq i \leq m$;

$f(v_i) \cup f(u_1) = f(v_i) \in f(V), 1 \leq i \leq m$;

$f^{\oplus}(u_1u_2) = \{1\} \in f(V), f^{\oplus}(u_2u_3) = \{1,2\} \in f^{\oplus}(E)$

and $f^{\oplus}(u_3u_4) = \{1,3\} \in f^{\oplus}(E)$.

Therefore, $f(V) \cup f^{\oplus}(E)$ forms a topology on X.

Hence$P_4 \odot K_{1,m}$is topogenic for every positive integer m.

Theorem 4

$P_5 \odot K_{1,m}$ is topogenic for every positive integer m.

Proof

Let $V(K_{1,m}) = \{u_1, v_1, v_2, \dots, v_m\}$ and

$V(P_4) = \{u_1, u_2, u_3, u_4, u_5\}$.

Then $V(P_5 \odot K_{1,m}) = V(P_5) \cup V(K_{1,m})$.

Without loss of generality, choose $X = \{1,2, \dots, m+3\}$ as the ground set.

Let $f: V(P_5 \odot K_{1,m}) \rightarrow 2^X$be defined by

$$f(v_i) = \{1,2,..i+3\}, 1 \leq i \leq m,$$

$f(u_1) = \emptyset, f(u_2) = \{1\}, f(u_3) = \{2\}, f(u_4) = \{1,3\}$,

$f(u_4) = \{1,2,3\}$.

Then, both f and $f^{\oplus}$ are injective.

$f(v_i) \cap f(u_1) = \emptyset \in f(V), 1 \leq i \leq m$;

$f^{\oplus}(u_1u_2) = \{1\} \in f(V), f^{\oplus}(u_2u_3) = \{1,2\} \in f^{\oplus}(E)$,

$f^{\oplus}(u_3u_4) = \{1,2,3\} \in f(V)$ and

$f^{\oplus}(u_4u_5) = \{2\} \in f(V)$.

Therefore, $f(V) \cup f^{\oplus}(E)$ forms a topology on X.

Hence $P_5 \odot K_{1,m}$ is topogenic for every positive integer m.

Corollary 5

Similarly, we can prove $P_n \odot K_{1,m}$, for

$n = 6,7,8,9,10,11,12,13,14$ is topogenic from the argument of P_n, for $n = 6,7,8,9,10,11,12,13,14$ is topogenic.

Hence $P_n \odot K_{1,m}$, for $n = 6,7,8,9,10,11,12,$

13,14 is topogenic.

Problem 6

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Is $P_n \odot K_{1,m}$ is topogenic for every positive integers m and $n \geq 15$.

Definition 7

We identify the central vertex of $K_{1,p}$ and the central vertex of $K_{1,m,n}$. The resulting graph is denoted by $K_{1,p} \odot K_{1,m,n}$.

Theorem 8

$K_{1,p} \odot K_{1,m,n}$ is topogenic for every positive integer m, n and p.

Proof

Let $V(K_{1,p}) = \{u_0, u_1, u_2, \dots, u_p\}$ and

$$V(K_{1,m,n}) = \{u_0, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\}.$$

Then $V\left((K_{1,p}) \odot (K_{1,m,n})\right) = V(K_{1,p}) \odot V(K_{1,m,n})$

Without loss of generality, choose

$X = \{1,2, \dots, m+n+p\}$ as the ground set.

Let $f: V(K_{1,p} \odot K_{1,m,n}) \to 2^X$ be defined by

$f(u_i) = \{1,2, \dots, i\}, m+n+1 \leq i \leq p, f(u_0) = \emptyset$.

Also, $f(v_i) = \{1,2, \dots, i\}, 1 \leq i \leq m$;

$f(w_j) = \{1,2, \dots, m+j\}, 1 \leq j \leq n$.

Let $f^{\oplus}$ be the induced edge function.

Then both f and $f^{\oplus}$ are injective.

Therefore, $f(V) \cup f^{\oplus}(E)$ forms a topology on X.

Hence $K_{1,p} \odot K_{1,m,n}$ is topogenic for every positive integer m, n and p.

Example 9

If the topogenic graph G is connected, then the corresponding topology is either connected or not connected.

For,

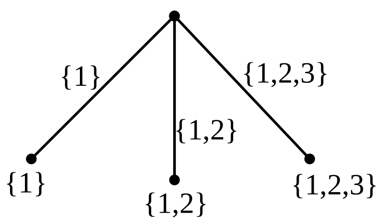

Figure 1.1

$\tau = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$.

Therefore, (X, τ) is connected.

Then , the next example shows that if the graph is connected, then the topology is not necessarily connected.

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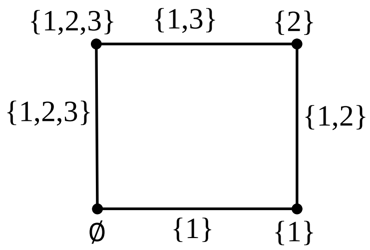

Figure 1.2

$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$.

Therefore, (X, τ) is not connected.

Example 10

If the topogenic graph G is disconnected, then the topology is either connected or not connected.

For,

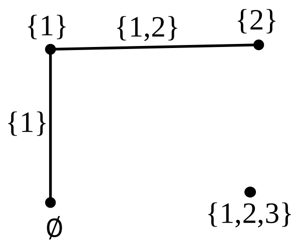

Figure 1.3

$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$.

Therefore, (X, τ) is connected.

Then, the next example show that if the graph is disconnected, then the topology is not connected.

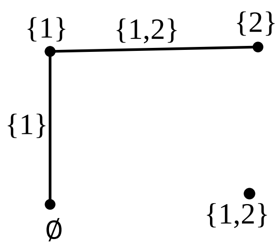

Figure 1.4

$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2\}\}$.

Therefore, (X, τ) is not connected.

Remark 11

From the above examples we conclude that the graph C_3 and C_4 are topogenic.

Proposition 12

C_5 is not topogenic.

Proof

If possible, let C_5 be topogenic, which implies there exists a topogenic set-indexer f of C_5 with respect to some non-empty ground set, say X, so that

$\tau_f = f(V(C_5)) \cup f^{\oplus}(E(C_5))$ is a topology on X.

Then by Theorem 3.11, for a topogenic cycle C_n,

$$n + 1 \leq \rho^0 \leq 2n - 2.$$

For C_5, $6 \leq \rho^0 \leq 8$.

Since f is injective, the empty set, $\emptyset$ cannot be obtained as a symmetric difference of two non-empty sets.

Hence, empty set, $\emptyset$ should necessarily be assigned to a vertex.

That is, $\emptyset \in f(V(C_5))$.

Hence, let $f(V(C_5)) = \{\emptyset, V_1, V_2, V_3, V_4\}$, where V_1, V_2, V_3, V_4 are non-empty subsets of X.

Then, $f^{\oplus}(E(C_5)) = \{V_1 \oplus \emptyset, V_1 \oplus V_2, V_2 \oplus V_3, V_3 \oplus V_4, V_4 \oplus V_5\}$.

Since τ_f is a topology on X, the entire set X must be an element of τ_f.

There arise two cases namely, $X = V_i$ for some $i \in \{1,2,3,4\}$, or $X = V_i \oplus V_j$ for some distinct $i, j \in \{1,2,3,4\}$.

Case (i): $X = V_i$ for some i.

Step 1

Without loss of generality, let $X = V_4$.

Then V_1, V_2, V_3 can be such that $V_1 \cup V_2 \cup V_3 = V_4$ or

$V_1 \cup V_2 \cup V_3 \subset V_4$.

Let $V_1 \cup V_2 \cup V_3 = V_4$.

If the sets, V_1, V_2, V_3 are pairwise disjoint, then

$V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) = V_1 \cup V_2$ and

$$V_4 \oplus V_3 = (V_4 \cup V_3) \backslash (V_4 \cap V_3)$$

$= V_4 \backslash V_3$

$= X \backslash V_3$

$= V_1 \cup V_2$, a contradiction to the injectivity of $f^{\oplus}$.

Therefore, atleast two of the sets V_1, V_2, V_3 must have a common element.

Without loss of generality, suppose $V_1 \cap V_2 = A \neq \emptyset$.

Since τ_f is a topology on X, we have A must be in τ_f.

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Therefore, $A = V_i$ for some $i \in \{1,2,3\}$ or $A = V_i \oplus V_j$ for some distinct $i, j \in \{1,2,3\}$, for neither $A = V_4 = X$ nor $A = V_4 \oplus \emptyset$.

So let $A = V_3$.

Then $V_1 \cup V_2 \cup V_3 = V_1 \cup V_2 \cup A = V_1 \cup V_2 = V_4$.

Then $V_1 \oplus V_2 = (V_1 \cup V_2) \setminus (V_1 \cap V_2) = V_4 \setminus A = V_1 \cup V_2$.

$$V_3 \oplus V_4 = (V_3 \cup V_4) \setminus (V_3 \cap V_4) = V_4 \setminus V_3 = X \setminus A$$
$$= V_1 \cup V_2.$$

Which is a contradiction to the injectivity of $f^{\oplus}$.

Therefore, $A = V_1$ or $A = V_2$.

Let $A = V_1$.Then $V_1 \cup V_2 = V_2$.

Now, $V_2 \cap V_3 \neq \emptyset$. Since otherwise, if $V_2 \cap V_3 = \emptyset$, then

$V_1 \cup V_2 \cup V_3 = V_2 \cup V_3$ and

by assumption $V_2 \cup V_3 = V_4$.

Then, $V_2 \oplus V_3 = (V_2 \cup V_3) \setminus (V_2 \cap V_3) = V_2 \cup V_3 = V_4$

But $V_4 \oplus \emptyset = V_4$, a contradiction to injectivity of $f^{\oplus}$.

Therefore, $V_2 \cap V_3 \neq \emptyset$.

Also, $V_1 \oplus V_2 = (V_1 \cup V_2) \setminus (V_1 \cap V_2) = V_2 \setminus V_1$

and $V_3 \oplus V_4 = (V_3 \cup V_4) \setminus (V_3 \cap V_4) = V_4 \setminus V_3$

$$= (V_1 \cup V_2 \cup V_3) \setminus V_3$$
$$= (V_1 \setminus V_3) \cup (V_2 \setminus V_3) \cup (V_3 \setminus V_3)$$
$$= (V_1 \setminus V_3) \cup (V_2 \setminus V_3)$$
$$= (V_1 \setminus V_2) \setminus V_3$$
$$= V_2 \setminus V_3$$

But by the choices of V_1, V_2, V_3, V_4 we have a contradiction to the injectivity of $f^{\oplus}$.

Hence $A \neq V_1$.

Analogously, we can show that $A \neq V_2$.

That is, $A \neq V_i$, for all $i \in \{1,2,3,4\}$.

Step 2

Hence A being an element of τ_f, $A = V_i \oplus V_j$ for some $i, j \in \{1,2,3,4\}$.

But $V_1 \oplus V_2 = (V_1 \cup V_2) \setminus (V_1 \cap V_2) = (V_1 \cup V_2) \setminus A$.

$V_2 \oplus V_3 = (V_2 \cup V_3) \setminus (V_2 \cap V_3)$.

$$V_3 \oplus V_4 = (V_3 \cup V_4) \setminus (V_3 \cap V_4) = V_4 \setminus V_3 = (V_1 \cup V_2) \setminus V_3$$

and hence none of the sets $V_i \oplus V_j$ for distinct

$i, j \in \{1,2,3,4\}$ equals A.

Hence $A = V_1 \cap V_2 \notin \tau_f$, again a contradiction to the fact that τ_f is a topology on X.

The above analysis implies that $V_1 \cup V_2 \cup V_3 \neq X$.

Then $V_1 \cup V_2 \cup V_3 \subset V_4 = X$.

Let $V_1 \cup V_2 \cup V_3 = B$.

Since τ_f being a topology, B must be in τ_f.

Clearly $B \not\subseteq V_i$ and $B \neq V_i$ for any $i \in \{1,2,3,4\}$,and not a subset of the union of any pair of them.

Hence $B = V_i \oplus V_j$ for some distinct $i, j \in \{1,2,3,4\}$.

Without loss of generality, let $B = V_1 \oplus V_2$.

Then $V_1 \cup V_2 \cup V_3 = B = V_1 \oplus V_2$.

Then V_1, V_2 can be such that $V_1 \cup V_2 \subset B$ or $V_1 \cup V_2 = B$

Suppose $V_1 \cup V_2 \subset B$.

Then $V_1 \cap V_2 = \emptyset$ which implies

$$V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) = (V_1 \cup V_2) \subset B$$

and

$V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) \subset B \backslash (V_1 \cup V_2) \subset B$.

Suppose, $V_1 \cup V_2 = B$.

Then $V_1 \cap V_2 \neq \emptyset$ which implies

$V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) = B \backslash (V_1 \cup V_2) \subset B$ and $V_1 \cap V_2 \neq \emptyset$ which implies suppose V_1 and V_3 are disjoint sets, then $V_1 \in \tau_f$ and $V_3 \in \tau_f$.

Since τ_f is a topology, $V_1 \cup V_3 \in \tau_f$.

By our choices of V_1, V_2, V_3 and V_4 and from the expressions for $V_i \oplus V_j$ for distinct $i, j \in \{1,2,3,4\}$, it is clear that $V_1 \cup V_3 \notin \tau_f$,this leads to a contradiction.

Therefore, V_1 and V_3 must have a common element.

Therefore, there exists D such that $D = V_1 \cap V_3$ and $D \neq \emptyset$.

Which implies $D = V_1 \cap V_3 \subset B$

Which implies $D \in \tau_f$.

By our choices of V_1, V_2, V_3 and V_4 can be such that $V_4 \neq D$.

Therefore, $D = V_1$ or $D = V_2$ or $= V_3$.

Suppose $D \subseteq V_2$.

$D \subseteq V_1 \cap V_2$.

Which is a contradiction.

Therefore, $D = V_1$ or $D = V_2$.

Suppose $D = V_3$.

Then $V_1 \cap V_3 = V_3$.

Then $V_1 \oplus V_2 = (V_1 \cup V_2) \backslash (V_1 \cap V_2) = V_1 \cup V_2 \in \tau_f$.

$$V_3 \oplus V_4 = (V_3 \cup V_4) \backslash (V_3 \cap V_4) = V_4 \backslash V_3 \in \tau_f.$$

Since $V_1 \oplus V_2, V_3 \oplus V_4 \in f^{\oplus}(E(C_5)) \in \tau_f$.

But by choices of V_1, V_2, V_3 and from the expression for $V_i \oplus V_j$ for distinct $i, j \in \{1,2,3,4\}$, it is clear that $(V_1 \oplus V_2) \cap (V_3 \oplus V_4) \notin \tau_f$.

This leads to a contradiction to $V_1 \cap V_3 = V_1$.

Therefore, $V_1 \cap V_2 \neq \emptyset$.

In all the cases, $V_1 \oplus V_2 \subset B$.

That is $V_1 \cup V_2 \cup V_3 = B = V_1 \oplus V_2 \subset B$.

Which is impossible.

Thus, it follows that $X \neq V_i$, for all $i \in \{1,2,3,4\}$.

Case (ii)

Let $X = V_i \oplus V_j$, for some distinct $i, j \in \{1,2,3,4\}$.

Without loss of generality, assume that $X = V_1 \oplus V_2$.

Then $V_1 \cup V_2 = X$ and $V_1 \cap V_2 = \emptyset$, for if $V_1 \cap V_2 \neq \emptyset$, then $V_1 \oplus V_2 = (V_1 \cup V_2)\backslash(V_1 \cap V_2) \neq X$.

Then, since for every $i \in \{1,2,3,4\}$; $V_i \subset X$, V_3 and V_4 have non-empty intersection with atleast one of the sets $V_1$and V_2.

Without loss of generality, assume that

$C = V_2 \cap V_3 \neq \emptyset$.

Then C must be in τ_f and $C \neq \emptyset, V_1$.

But none of the sets $V_i \oplus V_j$ for distinct $i, j \in \{1,2,3,4\}$ can be the set C.

Therefore, C should necessarily be V_2, V_3 and V_4.

Now, let $C = V_4$.It can be shown that $V_2 \cup V_3$ cannot be equal to V_i,for all$i \in \{1,2,3,4\}$ and also $V_2 \cup V_3 \neq V_i \oplus V_j$ for all distinct $i, j \in \{1,2,3,4\}$.

Hence $C \neq V_4$.

Therefore, $C = V_2$ or V_3.

We claim that $C \neq V_2$ and $C \neq V_3$.

Suppose $C = V_3$, then $V_3 \subset V_2$.

Which implies $V_2 \cup V_3 = V_2$.

But $V_2 \cup V_3 \subset V_4$. (by our assumption)

Then $V_2 \subset V_4$ and hence $V_4 \backslash V_2 = K$, a non-empty subset of X.

Now, $V_2 \oplus V_4 = (V_2 \cup V_4)\backslash(V_2 \cap V_4) = V_4 \backslash V_2 = K$.

Since $K, V_3 \in \tau_f$, $K \cup V_3$ must be in τ_f. Since K is neither contained in V_2 nor in V_3 and $V_4 \neq K$ we get $K \cup V_3 \neq V_i$, for all $i \in \{1,2,3,4\}$.

Now,

$V_1 \oplus V_2 = (V_1 \cup V_2)\backslash(V_1 \cap V_2) = X \backslash \emptyset = X \neq K \cup V_3$.

$V_2 \oplus V_3 = (V_2 \cup V_3)\backslash(V_2 \cap V_3) = V_2 \backslash V_3 \neq K \cup V_3$.

$V_3 \oplus V_4 = (V_3 \cup V_4)\backslash(V_3 \cap V_4) = V_4 \backslash V_3 \neq K \cup V_3$.

Hence $\cup V_3 \neq V_i \oplus V_j$, for all $i, j \in \{1,2,3,4\}$.

That is $K \cup V_3 \notin \tau_f$, a contradiction to the fact that $\tau_f$is a topology.Hence $C \neq V_3$.

A similar contradiction aries when $C = V_2$.

Therefore $C \neq V_2$ and $C \neq V_3$.

Further, $V_2 \oplus V_3 = (V_2 \cup V_3)\backslash(V_2 \cap V_3) = (V_2 \cup V_3)\backslash C$

and since $V_2 \cup V_3 \subset V_4$,

$$V_1 \oplus V_2 = (V_1 \cup V_2)\backslash(V_1 \cap V_2) = X$$

and $V_3 \oplus V_4 = (V_3 \cup V_4)\backslash(V_3 \cap V_4) = V_4 \backslash V_3$.

We observe that $C \neq V_i$ for any $i \in \{1,2,3,4\}$and $\neq V_i \oplus V_j$, for all distinct $i, j \in \{1,2,3,4\}$.

This is again a contradiction to the fact that $C \in \tau_f$.

Hence $\neq V_i \oplus V_j$, for all distinct $i, j \in \{1,2,3,4\}$.

Therefore, C_5 is not topogenic.

Corollary 13

Similarly,we can prove that C_6 is not topogenic from the argument of C_5 is not topogenic.

Proposition 14

C_7 is topogenic.

Proof

Let $V(C_7) = \{v_1, v_2, v_3, \dots., v_7\}$.

Let $X = \{1,2,3\}$.

Define $f: V(C_7) \to 2^X$ such that

$$f(v_1) = \emptyset, f(v_2) = \{1\}, f(v_3) = \{2\}, f(v_4) = \{1,3\}$$

$f(v_5) = \{1,2,3\}, f(v_6) = \{1,2\}, f(v_7) = \{2,3\}$.

$f^{\oplus}(v_1v_2) = \{1\}, f^{\oplus}(v_2v_3) = \{1,2\}$,

$f^{\oplus}(v_3v_4) = \{1,2,3\}$, $f^{\oplus}(v_4v_5) = \{2\}$,

$f^{\oplus}(v_5v_6) = \{3\}$, $f^{\oplus}(v_6v_7) = \{1,3\}$.

Then $f(V((C_7)) \cup f^{\oplus}(E\big((C_7)\big) = \{\emptyset, \{1\}, \{2\}, \{3\},$

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} = 2^X$

Proposition 15

C_8 is topogenic.

Proof

Let $V(C_8) = \{v_1, v_2, v_3, \dots., v_8\}$.

Let $X = \{1,2,3,4\}$.

Define $f: V(C_8) \to 2^X$ such that

$$f(v_1) = \emptyset, f(v_2) = \{1,2,3,4\}, f(v_3) = \{1\},$$

$f(v_4) = \{2\}, f(v_5) = \{1,3\}, f(v_6) = \{1,2,3\}$,

$f(v_7) = \{1,2\}, f(v_8) = \{2,3\}$.

Then $f(V((C_8)) \cup f^{\oplus}(E\big((C_8)\big) = \{\emptyset, \{1\}, \{2\}, \{3\},$

$$\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

Therefore, C_8 is topogenic.

We propose the following problem: For further study open problem.

Problem 16

Is C_n topogenic, when$n \geq 9$.

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REFERENCES

1. B.D.Acharya, Set-indexers of graph and set-graceful graphs, Bull.Allahabad Math, Soc., 16(2001), 1-23.
2. K.A.Germina, Jisha Eizabath Joy, Topogenic Graphs, submitted.