International Journal of Psychosocial Rehabilitation, Vol. 24, Issue 02, 2020
ISSN: 1475-7192

Double Sequence Space of Fuzzy Real numbers Defined by Double Orlicz Function

[1]Ali Hussein Battor, [2]Alaa Fawzidabbas, [3]Maysoon Azeen Neamah

Abstract-----*Through this paper, we introduce some double sequence space of fuzzy real numbers defined by a double Orlicz function. We study some of their features such that solidness, symmetricity, completeness etc., and prove some Important results.*

Keywords*-----* *Double Orlicz function, regular convergence, completeness, symmetric space, solid space.*

---

1. Introduction

In our paper, $(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X}_{s,r}\,,\,\mathfrak{M}_{s,r})$,represents a double infinite array of elements$\mathfrak{X}_{s,r}\,,\mathfrak{M}_{s,r}$we mean that$\mathfrak{X} = (\mathfrak{X}_{s,r})$be an infinite array of elements$\mathfrak{X}_{s,r}$and that$\mathfrak{M} = (\mathfrak{M}_{s,r})$be an infinite$\mathfrak{X}_{s,r}\,,\,\mathfrak{M}_{s,r}$array of elements$\mathfrak{M}_{s,r}and\mathcal{W}^{\parallel}$ denotes the family of all$\mathcal{C}^2$double sequences (i.e., $(\mathfrak{X}_{s,r}\,,\,\mathfrak{M}_{s,r})$is complex double sequences).

The concept of fuzzy sets was presented by Zadeh [15]. It has wide range of implementation in almost all the branches of studies. It attracted workers on double sequence spaces to present by different type of classes of double sequences of fuzzy numbers.

Throughout the paper$(L_\infty)_F^{\parallel}, (c)_F^{\parallel}, (c_0)_F^{\parallel}, (c^R)_F^{\parallel}, (c_0^R)_F^{\parallel}$denote the classes of all bounded , convergent in pringsheim's sense, null in pringsheim's sense, regularly convergent, regularly null, convergent in pringsheim's sense of fuzzy real numbers respectively.

Now, we define the N-function $\Upsilon(\mathfrak{X},\mathfrak{M})$in theterm of double sequence spaces as follows:

Definition1.1[1,2]: A double Orlicz function is a function

$M: [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$Such that

$M(x,y) = (M_1(x), M_2(y))$,

$M_1: [0,\infty) \to [0,\infty)andM_2: [0,\infty) \to [0,\infty)$,

such that $M_1, M_2$are Orlicz function which are continuous, non-decreasing, even, convex and satisfy the following conditions

1) $M_1(0) = 0, M_2(0) = 0 \Longrightarrow M(0,0) = (M_1(0), M_2(0)) = (0,0)$,

2)$M_1(x) > 0, M_2(y) > 0 \Longrightarrow M(x,y) = (M_1(x), M_2(y)) > (0,0)$

---

[1] *Department of Mathematics, Faculty of Education for Girls University of Kufa , Najaf – Iraq, E-mail : alih.battoor@uokufa.edu.iq*
2Department of Mathematics, Faculty of Education for Girls University of Kufa , Najaf – Iraq, E-mail:lionwhite812@gmail.com
3Department of Mathematics, Faculty of Education for Girls University of Kufa , Najaf – Iraq, E-mail:Omfatima75@gmail.com

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

for $x > 0, y > 0$, we mean by $M(x,y) > (0,0)$, that $M_1(x) > 0, M_2(y) > 0$.

3)$M_1(x) \to \infty, M_2(y) \to \infty$ as$x, y \to \infty$, then

$M(x,y) = (M_1(x), M_2(y)) \to (\infty,\infty)as(x,y) \to (\infty,\infty)$,we mean by

$M(x,y) \longrightarrow (\infty,\infty)$, that $M_1(x) \to \infty, M_2(y) \to \infty$.

Definition1.2:A double sequence$(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\ ,\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$of fuzzy real numbers is said to be a double Cauchy sequence if forevery$\epsilon > 0$ there exists $N \in \mathcal{N}$ such that$\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\ ,\mathfrak{X}_{i,j}),(\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\ ,\mathfrak{M}_{i,j}\ )\right) < \epsilon$where $\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\ ,\mathfrak{X}_{i,j}\ ) < \epsilon$ and $\bar{d}(\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\ ,\mathfrak{M}_{i,j}\ ) < \epsilon$for all $i \geq \mathfrak{s} \geq N, j \geq \mathfrak{r} \geq N$, where

Definition1.3:A double sequence$(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\ ,\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ of fuzzy real numbers is said to be bounded, if the set $\{(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\ ,\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) : \mathfrak{s},\mathfrak{r} \in \mathcal{N}\}$ of fuzzy real numbers is bounded

Battor and Neamah**[1]** used theidea of Orlicz function to construct the sequence spaceL_{Υ},we will use that idea to construct adouble sequence space as follows:

$$L_{\Upsilon}^{\shortparallel} = (L_{\Upsilon_1}^{\shortparallel}, L_{\Upsilon_2}^{\shortparallel}) = \left\{(\mathfrak{X},\mathfrak{M}) \in \mathcal{W}^{\shortparallel} : \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\left\{\Upsilon_1\left(\frac{\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right) \vee \Upsilon_2\left(\frac{\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right)\right\} < \infty, \text{for} \mathit{some}\ \rho > 0\right\},$$

where for

$$L_{\Upsilon_1}^{\shortparallel} = \left\{\mathfrak{X} = \mathfrak{X}_{\mathfrak{s},\mathfrak{r}} \in \mathcal{W}^{\shortparallel} : \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\left\{\Upsilon_1\left(\frac{\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right)\right\} < \infty, \text{forsome}\ \rho > 0\right\},$$

$$L_{\Upsilon_2}^{\shortparallel} = \left\{\mathfrak{M} = \mathfrak{M}_{\mathfrak{s},\mathfrak{r}} \in \mathcal{W}^{\shortparallel} : \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\left\{\Upsilon_2\left(\frac{\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right)\right\} < \infty, \text{for some}\ \rho > 0\right\}.$$

The space $L_{\Upsilon}^{\shortparallel}$ with the norm

$$||(\mathfrak{X},\mathfrak{M})||_{\Upsilon} = inf\left\{\rho > 0, \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\left\{\Upsilon_1\left(\frac{\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right) \vee \Upsilon_2\left(\frac{\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right)\right\} \leq 1\right\},$$

where

$$||\mathfrak{X}||_{\Upsilon_1} = inf\left\{\rho > 0, \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\Upsilon_1\left(\frac{\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right) \leq 1\right\},$$

$$||\mathfrak{M}||_{\Upsilon_2} = inf\left\{\rho > 0, \sum_{\mathfrak{s}=1}^{\infty}\sum_{\mathfrak{r}=1}^{\infty}\Upsilon_2\left(\frac{\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called a double Orlicz of a double sequence space**.**

The space $L_{\Upsilon}^{\shortparallel}$is closely related to the space$L^{\wp}$, which is an a double Orlicz of a double sequence space with$\Upsilon(\mathfrak{X},\mathfrak{M}) = |(\mathfrak{X},\mathfrak{M})|^{\wp}$, for $1 \leq \wp < \infty$**.**

2. DEFINITION AND PRELIMINARES

Troughout this study, adouble sequence is denoted by $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}),(\ \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$a double infinite array of fuzzy real numbers.

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

The primary studies on double sequences may be found in Bromwich [**4**]. Thereafter, it was searched by Hardy [**5**], Moricz [**6**], Moricz and Rhoades [**7**], Tripathy [**8**], Tripathy and Sarma[**9,10**], Basarir and Sonalcan [**11**], and many others. Hardy [**5**] studied the notion of regular convergence for double sequence**.**

We refer to the set of all closed and bounded intervals$x = [x_1, x_2]$on therealline R by symbol $\mathcal{D}$**.**

For $x = [x_1, x_2] \in \mathcal{D}$, $y = [y_1, y_2] \in \mathcal{D}$ and $z = [z_1, z_2] \in \mathcal{D}$,$w = [w_1, w_2,] \in \mathcal{D}$, defined

$$d\big((x, y), (z, w)\big) = \max[(|x_1 - y_1|, |x_2 - y_2|), (|z_3 - w_3|, |z_4 - w_4|)].$$

It is recognized that $(\mathcal{D}, d)$ is a complete metric space **.**

The following information is taken from [**14**]

"A fuzzy number $\mathcal{H}$ is a fuzzy combination on the real axis, i.e., a mapping $\mathcal{H}: \mathcal{R} \to \mathcal{f}(= [0,1])$ associating each real number v with its membership rank $\mathcal{H}(v)$, satisfies the following conditions :

1) The mapping $\mathcal{H}$ is convex if $\mathcal{H}(v) \geq \mathcal{H}(s) \wedge \mathcal{H}(a) = min\{\mathcal{H}(s), \mathcal{H}(a)\}$, where s$< v < a$.

2) The mapping $\mathcal{H}$ is normal if there exists $v_0 \in \mathcal{R}$ such that $\mathcal{H}(v_0) = 1$,

3) The mapping $\mathcal{H}$ is upper-semi continuous if, for each $\epsilon > 0$, $\mathcal{H}^{-1}([0, c + \epsilon))$ is open in the usual topology of $\mathcal{R}$ for all $c \in \mathcal{f}$**.**

4) The closure of $\{v \in \mathcal{R} : \mathcal{H}(v) > 0\}$, denoted by $[\mathcal{H}]^0$, is compact .

5) The mapping $\mathcal{H}$ is called non-negative if $\mathcal{H}(v) = 0$, for all $v < 0$. The set of all non-negative fuzzy real numbers is denoted by $\mathcal{R}^*(\mathcal{f})$.

For 0<$\propto \leq 1$, The α-level set $[\mathcal{H}]^\propto$, of the fuzzy real number $\mathcal{H}$, defined by

$$[\mathcal{H}]^\propto = \{v \in \mathcal{R}: \mathcal{H}(v) \geq \alpha\}.$$

The set of all upper-semi-continuous, normal , convex fuzzy real numbers is denoted by $\mathcal{R}(\mathcal{f})$ and throughout the paper, by a fuzzy real number we mean that the number belongs to $\mathcal{R}(\mathcal{f})$" .

Let $\bar{d}: R^2(I) \times R^2(I) \to R$ be defined by

$$\bar{d}\big((x, y), (z, w)\big) = sup_{0 \leq \propto \leq 1} d\big(([x]^\propto, [y]^\propto), ([z]^\propto, [w]^\propto)\big),$$

Then,$\bar{d}$defines a metric on $R^2(I)$ and it is well-known that$\big(R(I), \bar{d}\big)$) is acomplete metric space (one may refer to puri and Ralescu**[13]).**

Let $\mathfrak{X} = (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})$, $\mathfrak{B} = (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be double sequence. A double sequences $(\mathfrak{X}, \mathfrak{B}) = \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big)$ of fuzzy real numbers be a convergent in pringsheim's sense to a fuzzy real numbers (ℓ_1, ℓ_2) if $\lim_{\mathfrak{s},\mathfrak{r}\to\infty} \mathfrak{X}_{\mathfrak{s},\mathfrak{r}} = \ell_1$ and $\lim_{\mathfrak{s},\mathfrak{r}\to\infty} \mathfrak{M}_{\mathfrak{s},\mathfrak{r}} = \ell_2$ exists, and consequently

$$\lim_{\mathfrak{s},\mathfrak{r}\to\infty} \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big) = (\ell_1, \ell_2),$$

where $\mathfrak{s}, \mathfrak{r}$ tend to ∞ if each independently to other .

Let $\mathfrak{X} = (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})$, $\mathfrak{B} = (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be double sequence. A double sequence $(\mathfrak{X}, \mathfrak{B}) = \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big)$ of fuzzy real numbers be a regularly converge if it converges in the pringsheim's sense and the below limits well be exist :

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{X}_{\mathfrak{s},\mathfrak{r}} = \ell_{\mathfrak{r}}\,(\mathfrak{r} = 1,2,3, \dots .),$$

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{M}_{\mathfrak{s},\mathfrak{r}} = s_{\mathfrak{r}}\,(r = 1,2,3, \dots .),$$

and

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$$\lim_{\mathfrak{r}\to\infty} \mathfrak{X}_{\mathfrak{s},\mathfrak{r}} = j_{\mathfrak{s}} \ (\mathfrak{s} = 1,2,3, \dots),$$

$$\lim_{\mathfrak{r}\to\infty} \mathfrak{M}_{\mathfrak{s},\mathfrak{r}} = i_{\mathfrak{s}} \ (\mathfrak{s} = 1,2,3, \dots),$$

therefore

$$\lim_{\mathfrak{s}\to\infty} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) = (\ell_{\mathfrak{r}}, s_{\mathfrak{r}}),$$

$$\lim_{\mathfrak{r}\to\infty} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) = (j_{\mathfrak{s}}, i_{\mathfrak{s}}).$$

If $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}),(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$ such that $|\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}| \le |\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}|$ and $|\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}| \le |\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}|$, and consequently $|\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}| \le |\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}|$ for all $\mathfrak{s}, \mathfrak{r} \in \mathcal{N}$, then a double sequence space $E_F^{\parallel}$ is said to be solid.

Let $K = \{(\mathfrak{s}_i, \mathfrak{r}_i): i \in \mathcal{N}; \mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_3 < \cdots$ and $\mathfrak{r}_1 < \mathfrak{r}_2 < \mathfrak{r}_3 < \cdots\} \subseteq \mathcal{N} \times \mathcal{N}$ and $E_F^{\parallel}$ be a fuzzy double sequence space. A K-step space of $E_F^{\parallel}$ is a double sequence space $\gamma_K^E = \{(\mathfrak{X}_{\mathfrak{s}_i,\mathfrak{r}_i}) \in w_F^{\parallel}: (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}\}$.

A canonical pre-image of a double sequence$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$is a double sequence $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$ defined as follows:

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) = \begin{cases} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), & \text{If}(\mathfrak{s}, \mathfrak{r}) \in K, \\ (\bar{0}, \bar{0}), & \text{otherwise.} \end{cases}$$

[14] we defined a canonical pre-image of step space γ_K^E is a compilation of canonical pre-images of all elements in γ_K^E.

If it includes the canonical pre-images of all its step spaces, a double sequence space $E_F^{\parallel}$ is said to be monotone.

If$(\mathfrak{X}_{\pi(\mathfrak{s})\pi(\mathfrak{r})}, \mathfrak{M}_{\pi(\mathfrak{s})\pi(\mathfrak{r})}) \in E_F^{\parallel}$, where π is a permutation of $\mathcal{N}$,implies $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$, then a double sequence space $E_F^{\parallel}$ is said to be symmetrical.

A fuzzy real-valued double sequence space $E_F^{\parallel}$ is said to be convergence free if $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$ whenever$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\parallel}$ and$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) = (\bar{0}, \bar{0})$ implies$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) = (\bar{0}, \bar{0})$.

We define the following classes of sequences**:**

$(L_\infty)_F^{\parallel}\Upsilon=$

$$\left\{ \begin{matrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\parallel}: sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho}\right) \right\} < \infty \\ \text{forsome } \rho > 0, \end{matrix} \right\},$$

where

$$(L_\infty)_F^{\parallel}\Upsilon_1 = \left\{ \begin{matrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\parallel}: sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho}\right) \right\} < \infty \\ \text{forsome } \rho > 0, \end{matrix} \right\},$$

and

$$(L_\infty)_F^{\parallel}\Upsilon_2 = \left\{ \begin{matrix} (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\parallel}: sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho}\right) \right\} < \infty \\ \text{forsome } \rho > 0, \end{matrix} \right\}.$$

$(c)_F^{\parallel}\Upsilon=$

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$$\left\{\begin{array}{c}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in\mathcal{W}_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\ell_1)}{\rho}\right)\vee\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\ell_2)}{\rho}\right)\right\}=0\\ \text{for some }\rho>0,\end{array}\right\},$$

where

$$(c)_F^{\Vert}\Upsilon_1=\left\{\begin{array}{c}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})\in w_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\ell_1)}{\rho}\right)\right\}=0\\ \text{forsome }\rho>0,\end{array}\right\}$$

and

$$(c)_F^{\Vert}\Upsilon_2=\left\{\begin{array}{c}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in\mathcal{W}_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\ell_2)}{\rho}\right)\right\}=0\\ \text{forsome }\rho>0,\end{array}\right\}.$$

$(c_0)_F^{\Vert}\Upsilon=$

$$\left\{\begin{array}{c}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in\mathcal{W}_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\bar{0})}{\rho}\right)\vee\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\bar{0})}{\rho}\right)\right\}=0\\ \text{forsome }\rho>0,\end{array}\right\},$$

where

$$(c_0)_F^{\Vert}\Upsilon_1\left\{\begin{array}{c}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})\in\mathcal{W}_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\bar{0})}{\rho}\right)\right\}=0\\ \text{forsome }\rho>0,\end{array}\right\},$$

and

$$(c_0)_F^{\Vert}\Upsilon_2\left\{\begin{array}{c}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in\mathcal{W}_F^{\Vert}: lim_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\bar{0})}{\rho}\right)\right\}=0\\ \text{forsome }\rho>0,\end{array}\right\}.$$

Moreover, we define the classes of double sequences $(c^R)_F^{\Vert}(\Upsilon)$, and $(c_0^R)_F^{\Vert}(\Upsilon)$as follows**.**

A double sequence $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in(c^R)_F^{\Vert}(\Upsilon)$ if $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})\in(c)_F^{\Vert}(\Upsilon)$and the next limits exist :

$$\lim_{\mathfrak{s}}\left\{\Upsilon\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}_{\mathfrak{r}})}{\rho}\right)\right\}=0,\qquad as\ \mathfrak{s}\to\infty,\forall\ \mathfrak{r}\in\mathcal{N},$$

$$\lim_{\mathfrak{r}}\left\{\Upsilon\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},s_{\mathfrak{s}})}{\rho}\right)\right\}=0,\qquad as\ \mathfrak{r}\to\infty,\forall\ \mathfrak{s}\in\ \mathcal{N},$$

$$\lim_{\mathfrak{s}}\left\{\Upsilon\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{r}})}{\rho}\right)\right\}=0,\qquad as\ \mathfrak{s}\to\infty,\forall\ \mathfrak{r}\in\mathcal{N},$$

$$\lim_{\mathfrak{r}}\left\{\Upsilon\left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},r_{\mathfrak{s}})}{\rho}\right)\right\}=0,\qquad as\ \mathfrak{r}\to\infty\ \forall\ \mathfrak{s}\in\mathcal{N},$$

therefore

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$\lim_{\mathfrak{s}} \Upsilon(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$=(0,0), $as\ \mathfrak{s} \to \infty$, for each $\mathfrak{r} \in \mathcal{N}$,

$\lim_{\mathfrak{r}} \Upsilon(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$=(0,0), $as\ \mathfrak{r} \to \infty$, for each $\mathfrak{s} \in \mathcal{N}$.

A double sequence $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c_0^R)_F^{\Vert}(\Upsilon)$, if

$\mathfrak{X} = \mathfrak{X}_{\mathfrak{r}} = s_{\mathfrak{s}} = \bar{0}$,for all$\mathfrak{s}, \mathfrak{r} \in \mathcal{N}$,

and

$\mathfrak{M} = \mathfrak{M}_{\mathfrak{r}} = r_{\mathfrak{s}} = \bar{0}$,for all$\mathfrak{s}, \mathfrak{r} \in \mathcal{N}$.

We define

$$(m)_F^{\Vert} = (c)_F^{\Vert}(\Upsilon) \ \cap \ (L_\infty)_F^{\Vert}(\Upsilon),$$

$$(m_0)_F^{\Vert} = (c_0)_F^{\Vert}(\Upsilon) \ \cap (L_\infty)_F^{\Vert}(\Upsilon).$$

3) Main Results

Theorem 3.1: The classes of double sequences $(L_\infty)_F^{\Vert}(\Upsilon)$, $(c^R)_F^{\Vert}(\Upsilon)$, $({c_0}^R)_F^{\Vert}(\Upsilon)$,$(m)_F^{\Vert}(\Upsilon)$, and $(m_0)_F^{\Vert}(\Upsilon)$are complete metric spaces with respect to the distance definedby

$f\big((\mathfrak{X}, \mathfrak{M}), (\mathfrak{B}, \mathfrak{Q})\big)$=

$$inf\left\{\rho > 0: sup_{\mathfrak{s},\mathfrak{r}}\left\{\left(\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \vee \left(\Upsilon_2\left(\frac{\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right)\right\} \le 1\right\}$$

where

$$f(\mathfrak{X}, \mathfrak{M}) = inf\left\{\rho > 0: sup_{\mathfrak{s},\mathfrak{r}}\left(\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \ \le 1\right\}$$

$$f(\mathfrak{B}, \mathfrak{Q}) = inf\left\{\rho > 0: sup_{\mathfrak{s},\mathfrak{r}}\left(\Upsilon_2\left(\frac{\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \ \le 1\right\}$$

Proof :

We prove it for the case $(L_\infty)_F^{\Vert}(\Upsilon)$ and the other cases can be established next similar techniques..

Let $(\mathfrak{X}^i), (\mathfrak{M}^i)$ be any Cauchy sequences in$(L_\infty)_F^{\Vert}(\Upsilon_1), (L_\infty)_F^{\Vert}(\Upsilon_2)$ respectively, hence $(\mathfrak{X}^i, \mathfrak{M}^i) = (\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}})$ be a double Cauchy sequence in$(L_\infty)_F^{\Vert}(\Upsilon)$.

Let $\epsilon > 0$, $x_0, r > 0$ be fixed. Then for each $\frac{\epsilon}{r x_0} > 0$, there exists a positive integer N such that $f_{\Upsilon_1}(\mathfrak{X}^i, \mathfrak{X}^j) < \frac{\epsilon}{r\mathfrak{x}_0}$ and $f_{\Upsilon_2}\left((\mathfrak{M}^i, \mathfrak{M}^j)\right) < \frac{\epsilon}{r\mathfrak{x}_0}$, for $i, j \ge N$, and consequently,

$f_{\Upsilon}\left((\mathfrak{X}^i, \mathfrak{X}^j), (\mathfrak{M}^i, \mathfrak{M}^j)\right) = \left(f_{\Upsilon_1}\left((\mathfrak{X}^i, \mathfrak{X}^j)\right), f_{\Upsilon_2}\left((\mathfrak{M}^i, \mathfrak{M}^j)\right)\right) < \frac{\epsilon}{r\mathfrak{x}_0}$,
for all $i, j \ \ge N$.

By definition of f, we obtain

$$inf\left\{\rho > 0: sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right\} \leq 1\right\}$$

Thus,

$$sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right\} \leq 1$$

for all $i, j \geq N$.

$$\Rightarrow sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\} \leq 1$$

for each $i, j \geq N$,

$$\Rightarrow \left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\} \leq 1$$

for each $\mathfrak{s}, \mathfrak{r} \geq 1$ and for all $i, j \geq N$.

Hence one can find $r > 0$ with $\Upsilon_1\left(\frac{rx_0}{2}\right) \geq 1$ and $\Upsilon_2\left(\frac{rx_0}{2}\right) \geq 1$, such that

$$\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right) \leq \Upsilon_1\left(\frac{rx_0}{2}\right) \text{and} \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right) \leq \Upsilon_2\left(\frac{rx_0}{2}\right)$$

Hence, $\Upsilon\left(\frac{rx_0}{2}, \frac{rx_0}{2}\right) = \left(\Upsilon_1\left(\frac{rx_0}{2}\right), \Upsilon_2\left(\frac{rx_0}{2}\right)\right) \geq (1,1)$, therefore,

$$\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right), \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{f_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\} \leq \left(\Upsilon_1\left(\frac{rx_0}{2}\right), \Upsilon_2\left(\frac{rx_0}{2}\right)\right).$$

This implies that

$\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{r\mathfrak{X}_0}{2} \cdot f_{\Upsilon_1}((\mathfrak{X}^i, \mathfrak{X}^j))$, for all $i, j \geq n_0$.

$$\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{r\mathfrak{X}_0}{2} \cdot \frac{\epsilon}{r\mathfrak{X}_0} = \frac{\epsilon}{2} \qquad \text{for all } i, j \geq n_0.$$

$$\Rightarrow \bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{\epsilon}{2} \qquad \text{for all } i, j \geq n_0.$$

and $\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{r\mathfrak{X}_0}{2} \cdot f_{\Upsilon_2}((\mathfrak{M}^i - \mathfrak{M}^j))$, for all $i, j \geq n_0$.

$$\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{r\mathfrak{X}_0}{2} \cdot \frac{\epsilon}{r\mathfrak{X}_0} = \frac{\epsilon}{2} \qquad \text{for all } i, j \geq n_0$$

$\Rightarrow \bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}}) \leq \frac{\epsilon}{2}$ for all $i, j \geq n_0$. then

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$\bar{d}\left((\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}), (\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})\right) \leq \frac{r\mathfrak{x}_0}{2} . \frac{\epsilon}{r\mathfrak{x}_0} = \frac{\epsilon}{2}$ for all $i, j \geq n_0$.

Hence $(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}})$ is a double Cauchy sequence in $R^2(\mathfrak{f})$.

Thus ,

For each $(0 < \epsilon < 1)$, there exists a positive integer N such that $\bar{d}\left((\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}), (\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M})\right) < \epsilon$ for all $i, j \geq N$. where $\bar{d}(\mathfrak{X}^i, \mathfrak{X}) < \epsilon$ and $\bar{d}(\mathfrak{M}^i, \mathfrak{M}) < \epsilon$ for all $i, j \geq N$.

Taking $j \to \infty$ and fixing i, so by using the continuity of $\Upsilon = (\Upsilon_1, \Upsilon_2)$, we get

$$sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}\left(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \lim_{j\to\infty} \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}\right)}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}\left(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \lim_{j\to\infty} \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}}\right)}{\rho} \right) \right\} \leq 1$$

Thus,

$$sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X})}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M})}{\rho} \right) \right\} \leq 1,$$

On taking the infimum of such ρ' s, we get,

$$inf \left\{ \rho > 0 : sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X})}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M})}{\rho} \right) \right\} \leq 1 \right\} \leq \epsilon$$

for all $i \geq N$ and $j \to \infty$.

Since $(\mathfrak{X}^i, \mathfrak{M}^i) \in (L_\infty)^{\Vert}_F(\Upsilon)$ and Υ is continuous, it follows that $(\mathfrak{X}, \mathfrak{M}) \in (L_\infty)^{\Vert}_F(\Upsilon)$.

This completes the proof of the theorem.

Property 3.2: The class of double sequences$(L_\infty)^{\Vert}_F(\Upsilon)$ is symmetric but the class of double sequences$(c)^{\Vert}_F(\Upsilon)$, $(c_0)^{\Vert}_F(\Upsilon)$, $(c_0^R)^{\Vert}_F(\Upsilon)$, $(c^R)^{\Vert}_F(\Upsilon)$,are not symmetric.

Proof: Noticeably the classof double sequences $(L_\infty)^{\Vert}_F(\Upsilon)$is symmetric. However, other the class of double sequences, could be indicated by the following example.

Example 3.1:: Let's say the class of double sequences $(c)^{\Vert}_F(\Upsilon)$. Consider

$\Upsilon(\mathfrak{X}, \mathfrak{M}) = (\mathfrak{X}, \mathfrak{M})$ and suppose the double sequence $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be defined by

$$(\mathfrak{X}_{1\mathfrak{r}}, \mathfrak{M}_{1\mathfrak{r}})(v) = \begin{cases} (v+1\,, v+1), & \text{for} -1 \leq v \leq 0; \\ (-v+1\,, -v+1), & \text{for} \quad 0 \leq v \leq 1; \\ (0,0), & \text{otherwise,} \end{cases}$$

where

$$(\mathfrak{X}_{1\mathfrak{r}})(v) = \begin{cases} (v+1), & \text{for} -1 \leq v \leq 0; \\ (-v+1), & \text{for} \quad 0 \leq v \leq 1; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(\mathfrak{M}_{1\mathfrak{r}})(v) = \begin{cases} (v+1), & \text{for} -1 \leq v \leq 0; \\ (-v+1), & \text{for} \quad 0 \leq v \leq 1; \\ 0, & \text{otherwise,} \end{cases}$$

For $\mathfrak{s} > 1$, we have

$$\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for} -2 \leq v \leq -1; \\ -v, & \text{for } -1 \leq v \leq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\left(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for} -2 \leq v \leq -1; \\ -v, & \text{for } -1 \leq v \leq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and consequently$\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)$can be defined as

$$\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} (v+2, v+2), & \text{for} -2 \leq v \leq -1; \\ (-v, -v), & \text{for } -1 \leq v \leq 0; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Let $\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}\right)$,$\left(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\right)$ be a rearrangement of $\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\right)$, $\left(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)$ respect which is defined by

$$\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{s}}\right)(v) = \begin{cases} v+1, & \text{for} -1 \leq v \leq 0; \\ -v+1, & \text{for } 0 \leq v \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left(\mathfrak{Q}_{\mathfrak{s},\mathfrak{s}}\right)(v) = \begin{cases} v+1, & \text{for} -1 \leq v \leq 0\ ; \\ -v+1, & \text{for } 0 \leq v \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{s}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{s}}\right)$can be defined by

$$\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{s}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{s}}\right)(v) = \begin{cases} (v+1, v+1), & \text{for} -1 \leq v \leq 0; \\ (-v+1, -v+1), & \text{for } 0 \leq v \leq 1; \\ (0,0), & \text{otherwise.} \end{cases}$$

and for$\mathfrak{s} \neq \mathfrak{r}$, we have

$$\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for} -2 \leq v \leq -1; \\ -v, & \text{for} -1 \leq v \leq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\left(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for} -2 \leq v \leq -1; \\ -v, & \text{for} -1 \leq v \leq 0; \\ 0, & \text{otherwise,} \end{cases}$$

therefore, $\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)$ can be defined by

$$\left(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \left\{\begin{matrix} (v+2\,, v+2), & \text{for } -2 \leq v \leq -1, \\ (-v\,, -v), & \text{for } -1 \leq v \leq -1, \\ (0\ \,, 0), & \text{otherwise.} \end{matrix}\right\}$$

Thus,

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c)_F^{\Vert}(\Upsilon)$ but $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \notin (c)_F^{\Vert}(\Upsilon)$.Hence $(c)_F^{\Vert}(\Upsilon)$ is not symmetric. In same sense, it can be indicated that other spaces of double sequences are not symmetric too.

Property 3.3: The classes of double sequences $(L_\infty)_F^{\Vert}(\Upsilon), (c_0)_F^{\Vert}(\Upsilon)$ and $(c_0^R)_F^{\Vert}(\Upsilon), (m_0)_F^{\Vert}(\Upsilon)$are solid.

Proof: Consider the class of double sequences$(L_\infty)_F^{\Vert}(\Upsilon)$.Let $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (L_\infty)_F^{\Vert}(\Upsilon)$and $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$be such that.

$$\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \bar{0}) \le \bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})$$

and

$$\bar{d}(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}, \bar{0}) \le \bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})$$

and consequently

$$\bar{d}((\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0})) \le \bar{d}((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0}))$$

as$\Upsilon = (\Upsilon_1, \Upsilon_2)$ is increasing , we have

$$sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}\left((\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0})\right)}{\rho} \right) \right\} \le sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_2 \left(\frac{\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0})\right)}{\rho} \right) \right\}$$

Hence, the classes of double sequences$(L_\infty)_F^{\Vert}(\Upsilon)$ is solid. In same way, we could recognize other spaces are solid too by following same sense .

Property 3.4: The classes of double sequences$(c)_F^{\Vert}(\Upsilon), (c^R)_F^{\Vert}(\Upsilon)$and $(m)_F^{\Vert}(\Upsilon)$are not monotone and hence not solid.

Proof : The following Example will lead to such result.

Example 3.2: Consider the class of double sequences space $(c)_F^{\Vert}(\Upsilon)$ and Suppose $\Upsilon(\mathfrak{X}, \mathfrak{M}) = (\mathfrak{X}, \mathfrak{M})$. Let $J = \{(\mathfrak{s}, \mathfrak{r}) : \mathfrak{s} \ge \mathfrak{r}\} \subseteq \mathcal{N} \times \mathcal{N}$. Let $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be define as :

$$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{ \begin{array}{ll} (v+3), & \text{for} -3 \le v \le -2, \\ \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}, & \text{for} -2 \le v \le -1 + \mathfrak{s}^{-1}, \\ 0, & \text{otherwise.} \end{array} \right\}$$

and

$$(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{ \begin{array}{ll} (v+3), & \text{for} -3 \le v \le -2\ , \\ \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}, & \text{for} -2 \le v \le -1 + \mathfrak{s}^{-1}, \\ 0, & \text{otherwise.} \end{array} \right\}$$

and consequently

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v)$=

$$\left\{ \begin{array}{ll} (v+3, v+3), & \text{for} -3 \le v \le -2, \\ (\mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}), \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}), & \text{for} -2 \le v \le -1 + \mathfrak{s}^{-1}, \\ (0, 0), & \text{otherwise.} \end{array} \right\}$$

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

Then $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c)_F^{\|}(\Upsilon)$.

Let $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ be the canonical pre-image of $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})_J$ for the sub set J of $\mathcal{N} \times \mathcal{N}$ .Then

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) & \text{if } (\mathfrak{s},\mathfrak{r}) \in J, \\ \bar{0}, & \text{otherwise.} \end{Bmatrix}$$

and

$$(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) & \text{if } (\mathfrak{s},\mathfrak{r}) \in J, \\ \bar{0}, & \text{otherwise.} \end{Bmatrix}.$$

and consequently

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) & \text{if } (\mathfrak{s},\mathfrak{r}) \in J, \\ (\bar{0},\bar{0}) & \text{otherwise.} \end{Bmatrix}$$

Thus,$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \notin (c)_F^{\|}(\Upsilon)$. Hence, $(c)_F^{\|}(\Upsilon)$does not regard as a monotone. In the same way, It can be indicated that other spaces of double sequences are not monotone too.

Hence,the spaces $(c)_F^{\|}(\Upsilon), (c^R)_F^{\|}(\Upsilon)$ and $m(\Upsilon)$ are not solid.

corollary 3.5:$Z(\Upsilon_1) \cap Z(\Upsilon_2) \subseteq Z(\Upsilon_1 + \Upsilon_2)$, for $Z = (L_\infty)_F^{\|}(\Upsilon), (c)_F^{\|}(\Upsilon), (c_0)_F^{\|}(\Upsilon), (c^R)_F^{\|}(\Upsilon),, (c_0^R)_F^{\|}(\Upsilon)$.

Proof: It is a simple evidence, therefore we delete it**.**

corollary 3.*6*:Let$\Upsilon$and$\Upsilon_1$be two Orlicz function then $Z(\Upsilon_1) \subseteq Z(\Upsilon \circ \Upsilon_1)$ for $Z = (L_\infty)_F^{\|}, (c)_F^{\|}, (c_0)_F^{\|}, (c^R)_F^{\|}, (c_0^R)_F^{\|}(m)_F^{\|}$,and $(m_0)_F^{\|}$.

Proof: We prove the result for the case $Z = (c_0)_F^{\|}$, the other cases may be proved by following similar technique. Let $\epsilon > 0$, be given, there exists $n > 0$, such that $\epsilon = \Upsilon(n)$. Let $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in Z(\Upsilon_1)$, then, there exist $k_0, l_0 \in \mathcal{N}$, such that

$$\Upsilon_1 \left[\frac{\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0},\bar{0})\right)}{\rho} \right] < n, \text{for some } \rho > 0.$$

$$\text{Let}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) = \Upsilon_1 \left[\frac{\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0},\bar{0})\right)}{\rho} \right], \text{for some } \rho > 0.$$

Since Υ is continuous and non-decreasing, we get

$$\Upsilon(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) = \Upsilon \left[\Upsilon_1 \left[\frac{\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0},\bar{0})\right)}{\rho} \right] \right] < \Upsilon(n) = \epsilon,$$

for some $\rho > 0$.

Which implies that,$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in Z(\Upsilon \circ \Upsilon_1)$.

This completes the proof .

Corollary 3.7:$Z(\Upsilon) \subseteq (L_\infty)_F^{\|}(\Upsilon)$for $Z = (c^R)_F^{\|}, (c_0^R)_F^{\|}$.The inclusions are strict **.**

Proof: The inclusion $Z(\Upsilon) \subseteq (L_\infty)_F^{\|}(\Upsilon)$ for $Z = (c^R)_F^{\|}, (c_0^R)_F^{\|}$ is obvious. For establishing that the inclusions are proper, consider the following example.

Example 3.3 Let$\Upsilon(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X},\mathfrak{M})$. Let the double sequence$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$, be defined by for $\mathfrak{s} > r$,

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v)$=

$$\begin{cases} \left((\mathfrak{s}v-\mathfrak{s}-1)(\mathfrak{s}-1)^{-1},(\mathfrak{s}v-\mathfrak{s}-1)(\mathfrak{s}-1)^{-1}\right), & \text{for } 1+\mathfrak{s}^{-1} \le v \le 2; \\ (3-v,3-v), & \text{for } 2 \le v \le 3; \\ (0,0), & \text{otherwise.} \end{cases}$$

where

$$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{cases} \left((\mathfrak{s}v-\mathfrak{s}-1)(\mathfrak{s}-1)^{-1}\right), & \text{for } 1+\mathfrak{s}^{-1} \le v \le 2; \\ 3-v, & \text{for } 2 \le v \le 3; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{cases} \left((\mathfrak{s}v-\mathfrak{s}-1)(\mathfrak{s}-1)^{-1}\right), & \text{for } 1+\mathfrak{s}^{-1} \le v \le 2; \\ 3-v, & \text{for } 2 \le v \le 3; \\ 0, & \text{otherwise.} \end{cases}$$

and for $\mathfrak{s} \le \mathfrak{r}$,

$$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{cases} (\mathfrak{s}v-1)(\mathfrak{s}-1)^{-1}, & \text{for} \mathfrak{s}^{-1} \le v \le 1; \\ -v+2, & \text{for } 1 \le v \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{cases} (\mathfrak{s}v-1)(\mathfrak{s}-1)^{-1}, & \text{for} \mathfrak{s}^{-1} \le v \le 1; \\ -v+2, & \text{for } 1 \le v \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

and consequently$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$can be defined as

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v)$=

$$\begin{cases} \left((\mathfrak{s}v-1)(\mathfrak{s}-1)^{-1},(\mathfrak{s}v-1)(\mathfrak{s}-1)^{-1}\right), & \text{for} \mathfrak{s}^{-1} \le v \le 2; \\ (-v+2,-v+2) & \text{for } 1 \le v \le 2; \\ (0,0) & \text{otherwise.} \end{cases}$$

Then,$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (L_\infty)_F^{\Vert}(\Upsilon)$ but $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \notin Z(\Upsilon)$ for $Z = (c^R)_F^{\Vert}, (c_0^R)_F^{\Vert}$.

Property3.8 The class ofdouble sequences$(L_\infty)_F^{\Vert}(\Upsilon)$, $(c^R)_F^{\Vert}(\Upsilon)$, $(c_0^R)_F^{\Vert}(\Upsilon)$, $(c)_F^{\Vert}(\Upsilon)$, $(c_0)_F^{\Vert}(\Upsilon)$, $(m)_F^{\Vert}(\Upsilon)$, and $(m_0)_F^{\Vert}(\Upsilon)$ are not convergent free**.**

Proof: The result follows from the following example.

Example 3.4: Consider the classes of double sequences $(c)_F^{\Vert}(\Upsilon)$.

Let $\Upsilon(\mathfrak{X},\mathfrak{M})$= $(\mathfrak{X},\mathfrak{M})$ and$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ defined by$(\mathfrak{X}_{1\mathfrak{r}},\mathfrak{M}_{1\mathfrak{r}}) = (\bar{0},\bar{0})$and for other values,

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v)$=

$$\begin{cases} (1,1), & \text{for } 0 \le v \le 1; \\ \left(-\mathfrak{s}v(\mathfrak{s}+1)^{-1}+(2\mathfrak{s}+1)(\mathfrak{s}+1)^{-1},-\mathfrak{s}t(\mathfrak{s}+1)^{-1}+(2\mathfrak{s}+1)(\mathfrak{s}+1)^{-1}\right), & \text{for } 1 \le v \le 2+\mathfrak{s}^{-1}; \\ (0,0), & \text{otherwise.} \end{cases}$$

where

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

$$\left(\mathfrak{X}_{\mathfrak{s},\mathrm{r}}\right)(v) = \begin{cases} 1, & \text{for } 0 \le v \le 1\,; \\ -\mathfrak{s}v(\mathfrak{s}+1)^{-1} + (2\mathfrak{s}+1)(\mathfrak{s}+1)^{-1}, & \text{for } 1 \le v \le 2 + \mathfrak{s}^{-1}; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left(\mathfrak{M}_{\mathfrak{s},\mathrm{r}}\right)(v) = \begin{cases} 1, & \text{for } 0 \le v \le 1; \\ -\ \mathfrak{s}v(\mathfrak{s}+1)^{-1} + (2\mathfrak{s}+1)(\mathfrak{s}+1)^{-1}, & \text{for} 1 \le v \le 2 + \mathfrak{s}^{-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Let thedouble sequence $\left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}, \mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right)$, be defined by$(\mathfrak{B}_{1\mathrm{r}}, \mathfrak{Q}_{1\mathrm{r}}) = (\bar{0}, \bar{0})$, and for other values,

$$\left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}\right)(v) = \begin{cases} 1, & \text{for } 0 \le v \le 1; \\ (\mathfrak{s}-v)(\mathfrak{s}-v)^{-1}, & \text{for } 1 \le v \le \mathfrak{s}; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left(\mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right)(v) = \begin{cases} 1, & \text{for } 0 \le v \le 1; \\ (\mathfrak{s}-v)(\mathfrak{s}-v)^{-1}, & \text{for } 1 \le v \le \mathfrak{s}; \\ 0, & \text{otherwise.} \end{cases}$$

and consequently $\left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}, \mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right)$can be defined as

$$\left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}, \mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right)(v) = \begin{cases} (1,1), & \text{for } 0 \le v \le 1; \\ ((\mathfrak{s}-v)(\mathfrak{s}-1)^{-1}, (\mathfrak{s}-v)(\mathfrak{s}-v)^{-1}) & \text{for } 1 \le v \le \mathfrak{s}; \\ (0,0), & \text{otherwise.} \end{cases}$$

Then $\left(\mathfrak{X}_{\mathfrak{s},\mathrm{r}}\right) \in (c)_F^{\Vert}(\Upsilon)$, and $\left(\mathfrak{M}_{\mathfrak{s},\mathrm{r}}\right) \in (c)_F^{\Vert}(\Upsilon)$.

$\Longrightarrow \left(\mathfrak{X}_{\mathfrak{s},\mathrm{r}}, \mathfrak{M}_{\mathfrak{s},\mathrm{r}}\right) \in (c)_F^{\Vert}(\Upsilon)$ but$\left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}\right) \notin (c)_F^{\Vert}(\Upsilon)$, and $\left(\mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right) \notin (c)_F^{\Vert}(\Upsilon)$.

$$\Longrightarrow \left(\mathfrak{B}_{\mathfrak{s},\mathrm{r}}, \mathfrak{Q}_{\mathfrak{s},\mathrm{r}}\right) \notin (c)_F^{\Vert}(\Upsilon).$$

Hence, the classes of double sequences $(c)_F^{\Vert}(\Upsilon)$ is not convergent free. Similarly,

the other spaces are also not convergent free**.**

REFERENCES

[1] M. Azeez, (2017),**''On Statistically convergent double sequence spaces defined by Orlicz functions''**, Master Thesis, University of Kofa.

[2] Z. Hayder, (2017), **'' On Statistical convergent of generalized difference double sequence spaces which defined by Orlicz functions'',** Master Thesis, University of Kofa.

[3] Lindenstrauss J, and Tzafriri L, (1971),**''On Orlicz sequences spaces''** , Israel J. Math. 379-390.

[4] Bromwich, T. J.la. (1965),**''An Introduction to the Theory of Infinite Series'',** MacMillan and Co Ltd, New York.

[5] Hardy G .H (1917), **''On the convergence of certain multiple series'',** Proc. Camb Phil Soc,86-95.

[6] Moricz F (1991),**'' Extension of the spaces c and c_0 from single to double sequences'',**Acta Math Hungerica, 129-136 .

[7] Moricz F and Rhoades B E, (1988), **''Almost convergence of double sequences and strong regularity of summability matrices'',** Math. Proc. Camb Phil, Soc, 283-294.

[8] Tripathy B C, (2003), **''Statistically convergent double sequences'',**Tamkang J Math , 231-237.

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020

[9] Tripathy B C and Sarma, B, (2008),**''Statistically convergent difference double sequence spaces'',** Acta Math Sinica, 737-742.

[10] Tripathy B C, and Sarma B, (2009) **'' Vector valued double sequence spaces defined by Orlicz function'',** Math Slovaca , 767-776.

[11] Basarir M, and Sonalcan O ,(1999),**''On some double sequence spaces'',** J. Indian Acad . Math, 193-200.

[12] Kaleva O, and Seikkala S, (1984), "On **fuzzy metric spaces. Fuzzy Sets and Systems'',** 215-229.

[13] puri M I, and Ralescu D A, (1983), **''Differentials of fuzzy function'',** J Math Anal Appl, 552-558.

[14] Binod C, and Bipul S, (2011), **''Some double sequence spaces of fuzzy numbers defined by Orlicz function'',** Science Direct.

[15] L. A. Zadeh, (1965), **''Fuzzy sets'',** Information and Control, 338-353.

DOI: 10.37200/IJPR/V24I2/PR200696
Received: 02 Jan 2020 | Revised: 15 Jan 2020 | Accepted: 28 Jan 2020