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Common Fixed Point Theorems Of Gregus Type For Compatible Mappings In Banach Spaces

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Abstract
In this paper, we prove a common fixed point theorem of Gregus type for compatible mappings in Banach space. .Our work generalizes several earlier results on fixed points in this direction.

Key Words: *Common fixed point, compatible mappings, weakly compatible mappings, best approximant.*

AMS Subject classification (2000): Primary 54H25, Secondary 47H10

1 INTRODUCTION AND PRELIMINARIES:

The following definitions and results will be used in this paper.
In [8], Jungck defined the concept of compatibility of two mappings, which includes weakly commuting mappings (Sessa [15])as proper sub class.

1.1 Definition:
Let X be a normed linear space and let $S,T : X \to X$ be two mappings S and T are said to be compatible if, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to x \in X$, then

$$\|STx_n - TSx_n\| \to 0 \text{ as } n \to \infty$$

In (1998), Jungck and Rhoades[10] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true.

1.2 Definition:
A pair of S and T is called **weakly compatible** pair if they commute at coincidence points.

1.3 Example:
Consider $X = [0,2]$ with the usual metric d. Define mappings $S,T : X \to X$ by
$Sx = 0$ if $x = 0$, $Sx = 0.15$ if $x > 0$
$Tx = 0$ if $x = 0$, $Tx = 0.3$ if $0 < x \leq 0.5$, $Tx = x - 0.35$ if $x > 0.5$
Since S and T commute at coincidence point $0 \in X$, so S and T are weakly compatible maps to see that S and T are not compatible, let us consider a decreasing sequence $\{x_n\}$ where $x_n = 0.5 + \left(\frac{1}{n}\right), n = 1,2,...$ Then $Sx_n \to 0.15, Tx_n \to 0.15$ but $STx_n \to 0.15, TSx_n \to 0.3$ as $n \to \infty$. Thus weakly compatible compatible maps need not be compatible.

2. MAIN RESULTS:

We prove a common fixed point theorem of Gregus type for compatible mappings in Banach space. Our Theorem is improvement of results of Gregus[6], Jungck [9], Sharma and Deshpande [17].
Throughout this section, we assume that X is Banach space and C is non empty closed convex subset of X.
Now, we prove our main theorem.

2.1 Theorem :
Let S and T be compatible mappings of C into itself satisfying the following condition:

$$\|Tx - Ty\| \leq a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}$$

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$$+ d \max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\right\} \quad ....(1)$$

for all *x, y* in *C* where *a,b,c,d>0, a+b+c+d*=1 and $a+c+d<\sqrt{a}$ if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

Proof : Consider $x = x_0$ be an arbitarary point in *C* and choose points x_1, x_2 and x_3 in *C* such that
$Sx_1 = Tx$, $Sx_2 = Tx_1$, $Sx_3 = T x_2$
This can be done since $T(C) \subset S(C)$. for r = 1,2,3,... (1) leads to

$$\|Tx_r - Sx_r\| = \|Tx_r - Tx_{r-1}\|$$
$$\le a\|Sx_r - Sx_{r-1}\| + b\max\left\{\|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\right\}$$
$$+ c\max\left\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\right\}$$
$$+ d\max\left\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|, \frac{1}{2}\left(\|Tx_{r-1} - Sx_r\| + \|Tx_r - Sx_{r-1}\|\right)\right\}$$

which shows that,since

$$\|Sx_r - Sx_{r-1}\| = \|Tx_{r-1} - Sx_{r-1}\|,$$

we have, for *r* = 1,2,3,...

$$\|Tx_r - Sx_r\| \le \|Tx_{r-1} - Sx_{r-1}\|. \quad ....(2)$$

From (1) and (2) we have

$$\|Tx_2 - Sx_1\| = \|Tx_2 - Tx\|$$
$$\le a\|Sx_2 - Sx\| + b\max\left\{\|Tx_2 - Sx_2\|, \|Tx - Sx\|\right\}$$
$$+ c\max\left\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|\right\}$$
$$+ d\max\left\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|, \frac{1}{2}\left(\|Tx - Sx_2\| + \|Tx_2 - Sx\|\right)\right\}$$
$$\le a\|Tx_1 - Sx\| + b\max\left\{\|Tx - Sx\|, \|Tx - Sx\|\right\}$$
$$+ c\max\left\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|\right\}$$
$$+ d\max\left\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \frac{1}{2}\left(\|Tx - Tx_1\| + \|Tx_2 - Sx\|\right)\right\}$$
$$\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\left\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \frac{1}{2}\left(\|Tx - Sx_1\| + \|Tx_1 - Sx_1\| + \|Tx_2 - Sx_2\| + \|Sx_2 - Sx\|\right)\right\}$$
$$\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\left\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \frac{1}{2}\left(\|Tx - Sx\| + \|Tx - Sx\| + \|Tx_1 - Sx\|\right)\right\}$$
$$\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + 2d\|Tx - Sx\|$$
$$\le \left\{(2a + 2c + 2d) + b\right\}\|Tx - Sx\| \quad ....(3)$$

We shall now define a point

$$z = \left(\frac{1}{2}\right)x_2 + \left(\frac{1}{2}\right)x_3.$$

Since *C* is convex, $z \in C$ and *S* being linear

$$Sz = \left(\frac{1}{2}\right)Sx_2 + \left(\frac{1}{2}\right)Sx_3$$
$$= \left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 \quad ....(4)$$

It follows from (2), (3) and (4) that

$$\|Sz - Sx_1\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_1\right\|$$

$$\leq \left(\frac{1}{2}\right) \|Tx_1 - Sx_1\| + \left(\frac{1}{2}\right)\|Tx_2 - Sx_1\|$$

$$\leq \left(\frac{1}{2}\right) \|Tx - Sx\| + \left(\frac{1}{2}\right)\{(2a + 2c + 2d) + b\}\|Tx - Sx\|$$

$$\leq \left(\frac{1}{2}\right) \{1 + (2a + 2c + 2d) + b\}\|Tx - Sx\| \qquad ....(5)$$

By (2) and (4), we have

$$\|Sz - Sx_2\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_2\right\|$$

$$\leq \left(\frac{1}{2}\right) \|Tx_2 - Sx_2\|$$

$$\leq \left(\frac{1}{2}\right) \|Tx - Sx\|. \qquad ....(6)$$

By (1) and (6) we have

$$\|Tz - Sz\| = \left\|Tz - \left(\frac{1}{2}\right)Tx_1 - \left(\frac{1}{2}\right)Tx_2\right\|$$

$$\leq \left(\frac{1}{2}\right) \|Tz - Tx_1\| + \left(\frac{1}{2}\right)\|Tz - Tx_2\|$$

$$\leq \left(\frac{1}{2}\right) a\,\|Sz - Sx_1\| + \left(\frac{1}{2}\right) b \max\{\|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+ \left(\frac{1}{2}\right) c \max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+ \left(\frac{1}{2}\right) d \max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|,$$

$$\frac{1}{2}(\|Tx_1 - Sz\| + \|Tz - Sx_1\|)\} + \left(\frac{1}{2}\right) a\,\|Sz - Sx_2\| + \left(\frac{1}{2}\right) b \max\{\|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+ \left(\frac{1}{2}\right) c \max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+ \left(\frac{1}{2}\right) d \max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|,$$

$$\frac{1}{2}(\|Tx_2 - Sz\| + \|Tz - Sx_2\|)\}$$

$$\leq \left(\frac{1}{4}\right) a[1 + 2a + 2c + 2d + b]\|Tx - Sx\| + \left(\frac{1}{2}\right) b \max\{\|Tz - Sz\|, \|Tx - Sx\|\}$$

$$+ \left(\frac{1}{2}\right) c \max\left\{\frac{1}{2}(1 + 2a + 2c + b + 2d)\|Tx - Sx\|, \|Tz - Sz\|, \|Tx - Sx\|\right\}$$

$$+ \left(\frac{1}{2}\right) d \max\left[\frac{1}{2}(1 + 2a + 2c + b + 2d)\|Tx - Sx\|, \|Tz - Sz\|, \|Tx - Sx\|,\right.$$

$$\frac{1}{2}\{(2+2a+2c+2d+b)\|Tx-Sx\|+\|Tz-Sz\|\}\Big]+\left(\frac{1}{4}\right)a\|Tx-Sx\|$$
$$+\left(\frac{1}{2}\right)b\max\{\|Tz-Sz\|,\|Tx-Sx\|\}+\left(\frac{1}{2}\right)c\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|\right\} \qquad \leq \lambda\|Tx-Sx\| \text{ ….(7)}$$
$$+\left(\frac{1}{2}\right)d\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|,\ \frac{1}{2}(2\|Tx-Sx\|+\|Tz-Sz\|)\right\}$$

where

$$\lambda=\left(\frac{1}{4}\right)a[2+2a+2c+2d+b]+b+\frac{1}{4}c[1+2a+2c+b+2d]$$
$$+\frac{1}{2}c+\frac{1}{4}d[2+2a+2c+b+2d]+d$$
$$<\frac{1}{4}a(3+\sqrt{a})+\frac{1}{4}c(2+\sqrt{a})+b+\frac{1}{2}c+\frac{d}{4}(3+\sqrt{a})+d$$
$$<\frac{a}{4}+\frac{3a}{4}+b+c+d$$
$$=a+b+c+d=1$$

So we have 0< λ <1.

Since*x* is an arbitary point in *C*, from (7), it follows that there exists a sequence $\{z_n\}$ in C such that

$$\|Tz_0-Sz_0\|\leq\lambda\|Tx_0-Sx_0\|,$$
$$\|Tz_1-Sz_1\|\leq\lambda\|Tz_0-Sz_0\|,$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$\|Tz_n-Sz_n\|\leq\lambda\|Tz_{n-1}-Sz_{n-1}\|,$$

which yield that

$$\|Tz_n-Sz_n\|\leq\lambda^{n+1}\|Tx_0-Sx_0\|,$$

and so we have

$$\lim_{n\to\infty}\|Tz_n-Sz_n\|=0 \text{ ….(8)}$$

Setting $K_n=\left\{x\in C:\|Tx-Sx\|\leq\frac{1}{n}\right\}$

for *n* = 1,2, … then (8) shows that

$K_n\neq\phi$ for *n* = 1,2,…

and $K_1\supset K_2\supset K_3\supset\ldots$

obviously, we have $\overline{TK_n}\neq\phi$ and

$\overline{TK_n}\supset\overline{TK_{n+1}}$ for n = 1,2,…

for any *x, y* in K_n by (1), we have

$$\|Tx-Ty\|\leq a\|Sx-Sy\|+n^{-1}b+c\max\{\|Sx-Sy\|,n^{-1}\}$$
$$+d\max\left\{\|Sx-Sy\|,n^{-1},\ \frac{1}{2}(n^{-1}+\|Sx-Sy\|+n^{-1}+\|Sx-Sy\|)\right\}$$
$$\leq a\|Sx-Sy\|+n^{-1}b+c\max\{\|Sx-Sy\|,n^{-1}\}$$
$$+d\max\{\|Sx-Sy\|,n^{-1},\ (n^{-1}+\|Sx-Sy\|)\}$$
$$\leq a(2n^{-1}+\|Tx-Ty\|)+n^{-1}b+c(2n^{-1}+\|Tx-Ty\|)+d(3n^{-1}+\|Tx-Ty\|)$$
$$=[a+c]2n^{-1}+[a+c+d]\|Tx-Ty\|+n^{-1}b+3n^{-1}d$$

Therefore,

$$\|Tx - Ty\| \le n^{-1}\{2[a+c]+b+3d\}(1-a-c-d)^{-1}$$

Thus we have

$$\lim_{n\to\infty} diam\left(\overline{TK_n}\right) = \lim_{n\to\infty} diam\left(TK_n\right) = 0$$

By Cantor's theorem, there exists a point *u* in *C* such that

$$\bigcap_{n=1}^{\infty}\left(\overline{TK_n}\right) = \{u\}.$$

Since $u \in C$ for each $n = 1,2,\ldots$ there exists a point y_n in TK_n such that

$$\|y_n - u\| < n^{-1}$$

Then there exists a point x_n is K_n such that

$$\|u - Tx_n\| < n^{-1}$$

and so $Tx_n \to u \;\; as \;\; n \to \infty$.

Since $x_n \in k_n$, we have also

$$\|Tx_n - Sx_n\| < n^{-1}$$

and so $Sx_n \to u \;\; as \;\; n \to \infty$.

Since S is continuous $STx_n \to Su \; and \; SSx_n \to Su \;\; as \;\; n \to \infty$.

Moreover $\|TSx_n - STx_n\| \to 0 \; as \; n \to \infty$.

Since *S* and *T* are compatible and $Tx_n \to Sx_n \to u \;\; as \;\; n \to \infty$, we have $TSx_n \to Su$.

By (1), we have

$$\begin{aligned}\|Tu - Su\| &\le \|Tu - TSx_n\| + \|TSx_n - Su\| \\ &\le a\|Su - SSx_n\| + b\max\{\|Tu - Su\|, \|TSx_n - SSx_n\|\} \\ &\quad + c\max\{\|Su - SSx_n\|, \|Tu - Su\|, \|TSx_n - SSx_n\|\} \\ &\quad + d\max\{\|Su - SSx_n\|, \|Tu - Su\|, \|TSx_n - SSx_n\|, \\ &\quad \frac{1}{2}(\|TSx_n - Su\| + \|Tu - SSx_n\|)\} + \|TSx_n - Su\|\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\|Tu - Su\| &\le a\|Su - Su\| + b\max\{\|Tu - Su\|, \|Su - Su\|\} \\ &\quad + c\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|\} \\ &\quad + d\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|, \\ &\quad \frac{1}{2}(\|Su - Su\| + \|Tu - Su\|)\} + \|Su - Su\| \\ &= (b+c+d)\|Tu - Su\| \\ &= (1-a)\;\|Tu - Su\|.\end{aligned}$$

So we have *Tu* = *Su*.

Thus $TSu = STu$ and $TTu = TSu = STu$ since *S* and *T* are compatible. Furthermore, we have

$$\begin{aligned}\|TTu - Tu\| &\le a\|STu - Su\| + b\max\{\|TTu - STu\|, \|Tu - Su\|\} \\ &\quad + c\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\|\} \\ &\quad + d\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\| \\ &\quad \frac{1}{2}(\|Tu - STu\| + \|TTu - Su\|)\} \\ &= (a+c+d)\|TTu - Tu\|\end{aligned}$$

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This leads to $\|TTu - Tu\| = 0$ since $(a + c + d) < \sqrt{a}$.

Let $z = Tu = Su$.

Then $Tz = z$ and $Sz = STz = TSz = Tz = z$.

Thus *z* is a unique common fixed point of *T* and *S*. The uniqueness of *z* is a consequence of inequality (1). Now, we show that *T* is continuous at *z*. Let $\{y_n\}$ be a sequence in *C* such that $y_n \to z$.

Since *S* is continuous, $Sy_n \to Sz$, By (1), we have

$$\begin{aligned}
\|Ty_n - Tz\| &\le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Sy_n\|, \|Tz - Sz\|\} \\
&\quad + c\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|\} \\
&\quad + d\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|, \tfrac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\} \\
&\le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Tz\| + \|Tz - Sy_n\|\} \\
&\quad + c\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|\} \\
&\quad + d\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|, \|Tz - Sz\|, \\
&\qquad \tfrac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\} \\
&\le a\|Sy_n - Sz\| + b\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\} + c\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\} \\
&\quad + d\{\|Ty_n - Tz\| + \|Sz - Sy_n\|, \\
\|Ty_n - Tz\| &\le (a + b + c + d)\|Sy_n - Sz\| + (b + c + d)\|Ty_n - Tz\| \\
&\le (a + b + c + d)(1 - b - c - d)^{-1}\|Sy_n - Sz\|
\end{aligned}$$

Therefore, we have $Ty_n \to Tz$ and so T is continuous at *z*.

This completes the proof.

As a consequences of our Theorem 2.1, we have the following results.

2.2Corallary:

Let *S* and *T* be compatible mappings of *C* into itself satisfying the following condition:

$$\begin{aligned}
\|Tx - Ty\| &\le a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\} \\
&\quad + c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}
\end{aligned}$$

for all *x, y* in *C* where *a,b,c>0, a+b+c*=1 and *a+c*< $\sqrt{a}$ if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

Corallary 2.2 shows the result of Sharma and Deshpande [16], which obtain by putting *d* = 0 .

Now if b=0, c=0 then we get the following corallary

2.3Corallary:

Let *S* and *T* be compatible mappings of C into itself satisfying the following condition:

$$\|Tx - Ty\| \le a\|Sx - Sy\| + (1 - a)\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

for all *x, y* in *C*, 0 <a <1, if *S* is linear and continuous in *C* and $T(C) \subset S(C)$, Then *T* and *S* have a unique common fixed pointz in *C* and *T* is continuous at *z*.

2.4 Remark:

Corallary (2.3) also proves continuity of *T*, so it improves the result of Jungck[9].

if we put *a* = *b* = *c* = 0 then we get the following result

2.5 Corallary:

Let *S* and *T* be compatible mappings of *C* into itself satisfying the following condition:

$$\|Tx - Ty\| \le d\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \tfrac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\}$$

for all *x, y* in *C* where $0 \leq d < 1$, if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed pointz in *C* and *T* is continuous at *z*.
To demonstrate the validity of our Theorem 2.1, we have the following example

2.6 Example:

Let $X = R$ *and* $C = [0,1]$ with the usual norm. Consider the mappings T and S on C defined as $Tx = \frac{1}{4}x$ *and* $Sx = \frac{1}{2}x$ *for all* $x \in C$

Then $T(C) = \left[0, \frac{1}{4}\right] \subset S(C) = \left[0, \frac{1}{2}\right]$.

It is easy to see that S is linear and continuous.
Further, *T* and *S* are compatible if $\lim_{n\to\infty} x_n = 0$, *where* $\{x_n\}$ is a sequence in *C* such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$ for some $0 \in C$.

If we take $a = 1/9, b = 13/18, c = 3/18, d = 0$ we see that the condition (1) of our Theorem 3.1, is satisfied also we have *a* +*b*+*c*=1 and $a + c < \sqrt{a}$.
Thus all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of *S* and *T*.

REFERENCES

1. B. Brosowski., "Fixpunktsatze in der approximation-theory", Mathamatica (Cluj)., **11, (1969)**, 195.
2. Charbone., "Applications of Fixed Points to Approximation Theory", Jnanabha., **19, (1989)**, 63.
3. Carbone., "Applications of fixed point theorem", Jnanabha., **22, (1992),** 85
4. W.J. (Jr.) Dotson., "Fixed point theorems for non expansive mappings on starshaped Subsets of banach spaces", J. London. Math. Soc., **1 (1972)**, 408 – 410.
5. Fisher, and S. Sessa., "On a fixed point theorem of Gregus", Internet. J. Math. and Math. Sci., **9, (1986)**, 23.
6. M. Gregus "A fixed point theorem in Banach space", Boll. Um. Math. Ital., **(5), 17-A, (1980)**, 198.
7. T.L. Hicks, and M.D. Humphries, "A note on fixed point theorem", J. Approx. Theory, **34, (1986)**, 771.
8. Jungck, "Compatible mappings of Fisher and Sessa", Internat, J. Math. and Math., Sci., **9, (1986)**, 771.
9. G. Jungck, "On a fixed point theorem of Fisher and Sessa, Internal, J. Math and Math., Sci., **(1990)**, 497.
10. G. Jungck and B.E. Rhoades, Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math., **29(3) (1998)**, 227 – 238.
11. G. Meinardus., "Invariagebeilinearen approximation", Arch. Rational. Mech. Anal. **14 (1963)**, 301.303.
12. R.N. Mukherjee, and V. Verma, "A note on a fixed point theorem of Gregus, Math. Japonica, **33, (1988)**, 745.
13. H.K. Pathak, Y.J. Cho, and S.M. Kang, "An application of fixed point theorems, Internat, J. Math and Math., Sci., **21, (1998)**, 467.
14. S.A. Sahab, M.S. Khan, and S. Sess, "A result in best approximation theory", J. Approx. Theory, **55, (1988)**, 349.
15. S. Sessa., "On a weak commutativity conditions in fixed point consideration, Publ. Inst. Math. **46, (1982)**, 149.
16. Sushil Sharma, and Bhavna Deshpande., "Fixed point theorem and its application to best approximation theory", Bull. Cal, Soc., **93, (2) (2001)**, 155-166.
17. Sushil Sharma and Bhavana Deshpande., "Fixed point theorem for weakly compatible mappings and its application to best approximation theory", Journal of the Indian Math. Soc., Vol. 69, Nos. **1-4 (2002)**, 161-171.
18. M.L. Singh., "Application of fixed points to approximation theory", Proc. Approximation Theory and Applications., Pitman, London 198. **(1985)**.
19. S.P. Singh., "An applicationof a fixed point theorem to approximation theory, J. Approx. Theory., 25, **89 (1979)**.
20. P.V. Subrahmanyam., "An application of a fixed point theorem to best approximations", J. Approximation Theory., 20, **165 (1977).**