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Common Fixed Point Theorems For Weakly Compatible Mappings Satisfying Rational Contractive Conditions

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Abstract
By using notions of compatibility, weak compatibility and commutativity, we prove some common fixed point theorems for six mappings involving rational contractive conditions motivated by Nesic [13] in complete metric spaces. Our work generalizes some earlier results of Fisher [1], Jeong-Rhoades [6], Kannan [11] and others.

Keywords: *Complete metric spaces, fixed points, compatible mapping, weak compatible mapping.*

2000 AMS SUBJECT CLASSIFICATION CODE : 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES:

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's theorem and several others. Foremost among them is perhaps the weak commutativity condition introduced by Sessa [14] which can be described as follows:

1.1 Definition :
Let S and T be mappings of a metric space (X, d) into itself. Then (S, T) is said to be **weakly commuting** pair if $d(STx, TSx) \leq d(Tx, Sx)$ for all $x \in X$.
obviously, a commuting pair is weakly commuting, but its converse need not be true as is evident from the following example.

1.2 Example :

Consider the set X = [0, 1] with the usual metric. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$ for every $x \in X$. Then for all $x \in X$

$$STx = \frac{x}{4+2x}, \quad TSx = \frac{x}{4+x}$$

hence $ST \neq TS$. Thus$S$ and T do not commute.
Again

$$d(STx, TSx) = \left|\frac{x}{4+2x} - \frac{x}{4+x}\right| = \frac{x^2}{(4+x)(4+2x)}$$
$$\leq \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and so S and T commute weakly.
Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.

Jungck [8] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungek [8].

1.3 Definition :
Two self mappings$S$ and T of a metric space (X, d) are **compatible** if and only if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X. such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

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Clearly any weakly commuting pair {*S*,*T*} in compatible but the converse need not be true as can be seen in the following example.

1.4 Example :
Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then *S* and *T* are compatible, since

$|Tx - Sx| = |x^3| \to 0$ if and only if

$|STx - TSx| = 6|x^9| \to 0$ but

$|STx - TSx| \le |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

1.5 Proposition :
Let *S* and *T* be continuous self mapping on *X*. Then the pair (*S*, *T*) is compatible on X. where as in (Jungck [10], Gajic [4]) demonstrated by suitable examples that if *S* and *T* are discontinuous then the two concepts are independent of each other. The following examples also support this observation.

1.6 Example :
Let $X = R$ with the usual metric we define $S, T : X \longrightarrow X$ as follows.

$$Sx = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases} \quad and \quad Tx = \begin{cases} 1/x^3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Both *S* and *T* are discontinuous at $x = 0$ and for any sequence $\{x_n\}$ in *X*, we have $d(STx_n, TSx_n) = 0$. Hence the pair (S,T) is compatible.

1.7 Example :
Now we define

$$Sx = \begin{cases} 1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases} \quad and \quad Tx = \begin{cases} -1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

observe that the restriction of *S* and *T* on $(-\infty,1]$ are equal, thus we take a sequence $\{x_n\}$ in $(1, \infty)$. Then $\{Sx_n\} \subset (0,1)$ *and* $\{Tx_n\} \subset (-1,0)$. Thus for every *n*, $TTx_n = 0$, $TSx_n = 1$, $STx_n = 0$, $SSx_n = 1$. So that $d(STx_n, TTx_n) = 0$, $d(TSx_n, TTx_n) = 0$ for every $n \in N$. This shows that the pair (*S*, *T*) is compatible of type (A). Now let $x_n = n$, $n \in N$. Then $Tx_n \to 0,\ Sx_n \to 0\ \ as\ n \to \infty\ \ and\ STx_n = 0,\ TSx_n = 1$ for every $n \in N$ and so $d(STx_n, TSx_n) \neq 0$ as $n \to \infty$ hence the pair (S, T) is not compatible.

Very recently concept of **weakly compatible** obtained by Jungck-Rhoades [7] stated as the pair of mappings is said to be weakly compatible if they commute at their coincidence point.

1.8 Example :
Let X = [2, 20] with usual metric define

$$Tx = \begin{cases} 2 & if\ \ x = 2 \\ 12 + x & if\ \ 2 < x \le 5 \\ x - 3 & if\ \ 5 < x \le 20 \end{cases} \quad and \quad Sx = \begin{cases} 2 & if\ \ x \in \{2\} \cup (5,20] \\ 8 & if\ \ 2 < x \le 5 \end{cases}$$

S and *T* are weakly compatible mappings which is not compatible.

2. MAIN RESULTS :

Let R^+ be the set of non-negative real numbers and let $F: R^+ \longrightarrow R^+$ be a mapping such that *F*(0) and *F* is continuous at 0. The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [8].

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2.1 Lemma :
Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists a $k \in (0,1)$ such that $d(y_{n+1}, y_n) \le k\ (y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.
Motivated by the contractive condition given by, Jeong Rhoades [6] and Nesic [13] we prove the following theorem.

2.2 Theorem :
Let A,B,S,T,I and J be self mappings of a complete metric space (X,d) satisfying $AB(X) \subset J(X), ST(X), \subset I(X)$, and for each $x, y \in X$ either.

$$d(ABx, STy) \le \alpha_1 \left[\frac{d(ABx, Jy).d(Jy, STy) + d(STy, Ix)d(Ix, ABx)}{d(ABx, Jy) + d(STy, Ix)} \right] + \alpha_2 \left[d(ABx, Jx) + d(Jy, STy) \right] + \alpha_3 d(Ix, ABx) + F\left(d(STy, Ix).d(Ix, ABx) \right) \quad ...(1)$$

if $d(ABx, Jy) + d(STy, Ix) \ne 0, \alpha_1 \ge 0 (i = 1,2,3.....)$

with at least one α_1 non zero and $\alpha_1 + 2\alpha_2 + \alpha_3 \le 1$

$$d(ABx, STy) = 0 \; if \;\; d(ABx, Jy) + d(STy, Ix) = 0 \quad ....(2)$$

if either
(a) (AB, I) are compatible, I or AB is continuous and (ST,J) are weakly compatible or
(a') (ST,J) are compatible, J or ST is continuous then AB, ST, I and J have a unique common fixed point. Furthermore if the pairs $(A,B), (A,I), (B,I), (S,T), (S,J)$ and (T, J) are commuting mappings then A,B,S,T,I and J have a unique common fixed point.

Proof :
We construct the sequence as follows. Let x_0 be an arbitrary point in X. Since $AB(X) \subseteq J(X)$ we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Again since $ST(X) \subseteq I(X)$ we can choose a point x_2 in X. such that $STx_1 = Ix_2$, construct a sequence $\{z_n\}$ be repeateadly using this argument.

$z_{2n} = ABx_{2n} = Jx_{2n+1}, \qquad z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$ n = 0,1,2, ...

Let us put
$U_{2n} = d\ (ABx_{2n}, STx_{2n+1})$ and $U_{2n+1} = d\ (STx_{2n+1}, ABx_{2n+2})$ for n = 0,1,2,....

Now we distinguish to cases :
Case – 1 :
Suppose that $U_{2n} + U_{2n+1} \ne 0$ for n = 0,1,2..… Then on using inequality (1), we have

$$U_{2n+1} = d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$
$$\le \alpha_1 \left[\frac{d(ABx_{2n+2}, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + (STx_{2n+1}, Ix_{2n+2}).d(Ix_{2n+2}, ABx_{2n+2})}{d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})} \right]$$
$$+ \alpha_2 [d(ABx_{2n+2}, Jx_{2n+2}) + d(Jx_{2n+1}, STx_{2n+1})] + \alpha_3 d(Ix_{2n+2}, Jx_{2n+1})$$
$$+ F\left(d(STx_{2n+1}, Ix_{2n+2}).d(Ix_{2n+2}, ABx_{2n+2}) \right)$$

$$\le \alpha_1 \left[\frac{d(z_{2n+2}, z_{2n}).d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+1}).d(z_{2n+1}, z_{2n+2})}{d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n+1})} \right]$$
$$+ \alpha_2 [d(z_{2n+2}, z_{2n+1}) + d(z_{2n}, z_{2n+1})] + \alpha_3 d(z_{2n+1}, z_{2n})$$
$$+ F\left(d(z_{2n+1}, z_{2n+1}).d(z_{2n+1}, z_{2n+2}) \right)$$

$$= \alpha_1 d(z_{2n}, z_{2n+1}) + \alpha_3 d(z_{2n+2}, z_{2n+1}) + \alpha_2 d(z_{2n}, z_{2n+1}) + \alpha_3 d(z_{2n+1}, z_{2n}) + F(0)$$

which implies that

$$d(z_{2n+1}, z_{2n+2}) \le \frac{\alpha_1 + \alpha_2 + \alpha_3}{(1-\alpha_2)} d(z_{2n}, z_{2n+1})$$

Similarly we can conclude that

$$d(z_{2n}, z_{2n+1}) \le \frac{\alpha_1 + \alpha_2 + \alpha_3}{(1-\alpha_2)} d(z_{2n-1}, z_{2n})$$

Thus for every n we have

$$d(z_n, z_{n+1}) \le k d(z_{n-1}, z_n) \qquad ....(3)$$

where $k = \frac{\alpha_1 + \alpha_2 + \alpha_3}{(1-\alpha_2)} < 1$, therefore, by Lemma 2.1 $\{z_n\}$ converges to some $z \in X$. Hence this sequences.
$ABx_{2n} = Jx_{2n+1}$ and $STx_{2n+1} = Ix_{2n+2}$, which are subsequences also converges to the point z.

Let us now assume that I is continuous so that the sequence $\{I^2 x_{2n}\}$ and $\{IABx_{2n}\}$ converges to some point Iz. Also (AB,I) are compatible, so $\{ABIx_{2n}\}$ converges to Iz.
Now,

$$\begin{aligned} &d(ABI_{2n}, STx_{2n+1}) \\ &\le \alpha_1 \left[\frac{d(ABIx_{2n}, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + (STx_{2n+1}, I^2 x_{2n}).d(I^2 x_{2n}, ABIx_{2n})}{d(ABx_{2n}, Jx_{2n+1}) + d(STx_{2n+1}, I^2 x_{2n})} \right] \\ &+ \alpha_2 [d(ABIx_{2n}, I^2 x_{2n}) + d(Jx_{2n+1}, STx_{2n+1})] + \alpha_3 d(I^2 x_{2n}, Jx_{2n+1}) \\ &+ F\left(d(STx_{2n+1}, I^2 x_{2n}).d(I^2 x_{2n}, ABIx_{2n})\right) \end{aligned}$$

which on letting $n \to \infty$ reduces to

$$\begin{aligned} &d(Iz, z) \\ &\le \alpha_1 \left[\frac{d(Iz, z).d(z, z) + (Iz, z).d(Iz, Iz)}{d(Iz, z) + d(z, Iz)} \right] + \alpha_2 [d(Iz, Iz) + d(z, z)] \\ &+ \alpha_3 d(Iz, z) + F\left(d(Iz, z).d(Iz, Iz)\right) \end{aligned}$$

or

$$d(Iz, z) \le \alpha_3 d(Iz, z)$$

yielding thereby $Iz = z$.
Now,

$$\begin{aligned} &d(ABz, STx_{2n+1}) \\ &\le \alpha_1 \left[\frac{d(ABz, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + (STx_{2n+1}, Iz).d(Iz, ABz)}{d(ABz, Jx_{2n+1}) + d(STx_{2n+1}, Iz)} \right] \\ &+ \alpha_2 \left[d(ABz, Iz) + d(Jx_{2n+1}, STx_{2n+1})\right] + \alpha_3 d(Iz, Jx_{2n+1}) \\ &+ F\left(d(STx_{2n+1}, Iz) d(Iz, ABz)\right) \end{aligned}$$

Or

$$\begin{aligned} &d(ABz, z) \\ &\le \alpha_1 \left[\frac{d(ABz, z).d(z, z) + (z, Iz).d(Iz, ABz)}{d(ABz, z) + d(z, Iz)} \right] + \alpha_2 [d(ABz, Iz) + d(z, z)] \\ &+ \alpha_3 d(Iz, z) + F\left(d(z, Iz).d(Iz, ABz)\right), \end{aligned}$$

on letting $n \to \infty$ and using *Iz* =*z*, we get

$$d(ABz, z) \le \alpha_2 d(ABz, z)$$

implying thereby *ABz* = *z*.
Since $AB(X) \subset J(X)$, there always exist a point z' such that
Jz' = z so that *STz* = *ST* (Jz')

Now

$$\begin{aligned} d(z, STz') &= d(ABz, STz') \\ &\le \alpha_1 \left[\frac{d(ABz, Jz').d(Jz', STz') + d(STz') + d(STz', Iz).d(Iz, ABz)}{d(ABz, z') + d(STz', Iz)} \right] \\ &+ \alpha_2 [d(ABz, Iz) + d(Jz', STz')] + \alpha_3 d(Iz, Jz') + F(d(STz', Iz)d(Iz, ABz)) \end{aligned}$$

$$\begin{aligned} &= \alpha_1 \left[\frac{d(z,z).d(z, STz') + d(STz', z) + d(z,z)}{d(z,z) + d(STz', z)} \right] + \alpha_2 [d(z,z) + d(z, STz')] \\ &+ \alpha_3 d(z,z) + F(d(STz', z).d(z,z)) \\ &= \alpha_2 d(z, STz') + F(0) \end{aligned}$$

or

$$d(z, STz') \le \alpha_2 d(z, STz'),$$

which implies that $STz' = z = Jz'$. It shows that (*ST*,*J*) have a coincidence point z'. Now using the weak compatibility of (*ST*, *J*), we have

$$STz = ST(Jz') = J(STz') = Jz,$$

which shows that *z* is also a coincidence point of the pair (*ST*, *J*), Now

$$\begin{aligned} d(z, STz) &= d(ABz, STz) \\ &\le \alpha_1 \left[\frac{d(ABz, Jz).d(Jz, STz) + d(STz, Iz).d(Iz, ABz)}{d(ABz, Jz) + d(STz, Iz)} \right] \\ &+ \alpha_2 [d(ABz, Iz) + d(Jz, STz)] + \alpha_3 d(Iz, Jz) + F(d(ABz, Iz).d(Iz, STz)) \\ &= \alpha_1 \left[\frac{d(z, STz).d(STz, STz) + d(STz, z).d(z,z)}{d(z, STz) + d(STz, z)} \right] + \alpha_2 [d(z,z) + d(STz, STz)] \\ &+ \alpha_3 d(z, STz) + F(d(z,z).d(z, STz)) \\ &= \alpha_3 d(z, STz) + F(0) \end{aligned}$$

or

$$d(z, STz) \le \alpha_3 d(z, STz).$$

Hence $z = STz = Jz$, which shows that *z* is a common fixed point of *AB*, *I*,*ST* and *J*.

Now we suppose that *AB* is continuous so that the sequence $\{AB^2x_{2n}\}$ and $\{ABIx_{2n}\}$ converges to *ABz*. Since (*AB*, *I*) are compatible it follows that $\{IABx_n\}$ also converges to *ABz*. Thus

$$\begin{aligned} & d(AB^2 x_{2n}, STx_{2n+1}) \\ &\le \alpha_1 \left[\frac{d(AB^2 x_{2n}, Jx_{2n+1}).d(Jx_{2n+1}, STx_{2n+1}) + d(STx_{2n+1}, IABx_{2n+1}).d(IABx_{2n}, AB^2 x_{2n})}{d(AB^2 x_{2n}, Jx_{2n+1}) + d(STx_{2n+1}, IABx_{2n})} \right] \end{aligned}$$

$$+\alpha_2[d(AB^2x_{2n},IABx_{2n})+d(Jx_{2n+1},STx_{2n+1})]+\alpha_3 d(IABx_{2n},Jx_{2n+1})$$
$$+F\left(d(SIx_{2n+1},IABx_{2n}).d(IABx_{2n},AB^2x_{2n})\right)$$

which on letting $n\to\infty$ reduces to

$$d(ABz,z)$$
$$\le\alpha_1\left[\frac{d(ABz,z).d(z,z)+(z,ABz).d(ABz,ABz)}{d(ABz,z)+d(z,ABz)}\right]$$
$$+\alpha_2[d(ABz,ABz)+d(z,z)]+\alpha_3 d(ABz,z)$$
$$+F(d(z,ABz).d(ABz,ABz))$$

or

$$d(ABz,z)\le\alpha_3 d(ABz,z)$$

yielding thereby $ABz = z$

As earlier there exist z' in X such that $ABz = z = Jz'$. Then

$$d(AB^2x_{2n},STz')$$
$$\le\alpha_1\left[\frac{d(AB^2x_{2n},Jz').d(Jz',STz')+d(STz',IABx_{2n}).d(IABx_{2n},AB^2x_{2n})}{d(AB^2x_{2n},Jz')+d(STz',IABx_{2n})}\right]$$
$$+\alpha_2[d(AB^2x_{2n},IABx_{2n})+d(Jz',STz')]+\alpha_3 d(IABx_{2n},Jz')$$
$$+F\left(d(STz',IABx_{2n}).d(IABx_{2n},AB^2x_{2n})\right)$$

which on letting $n\to\infty$

$$d(z,STz')$$
$$\le\alpha_1\left[\frac{d(ABz,Jz').d(Jz',STz')+d(STz',ABz).d(ABz,ABz)}{d(ABz,Jz')+d(STz',ABz)}\right]$$
$$+\alpha_2[d(ABz,ABz)+d(Jz',STz')]+\alpha_3 d(ABz,Jz')$$
$$+F(d(STz',ABz).d(ABz,ABz))$$

or

$$d(z,STz')\le\alpha_2 d(z,STz')$$

This gives $STz'=z=Jz'$. Thus z' is a coincidence point of ST and J. Since the view of weakly compatibility of the pair (ST,J) one has $STz=ST(Jz')=J(STz')=Jz$ which shows that $STz = Jz$. Further

$$d(ABx_{2n},STz)$$
$$\le\alpha_1\left[\frac{d(ABx_{2n},Jz).d(Jz,STz)+d(STz,Ix_{2n}).d(Ix_{2n},ABx_{2n})}{d(ABx_{2n},Jz)+d(STz,Ix_{2n})}\right]$$
$$+\alpha_2[d(ABx_{2n},Ix_{2n})+d(STz,Jz)]+\alpha_3 d(Ix_{2n},Jz)$$
$$+F\left(d(Ix_{2n},STz).d(Ix_{2n},ABx_{2n})\right)$$

which on letting $n\to\infty$, and $STz = Jz$, reduces to

$$d(z,STz)$$
$$\le\alpha_1\left[\frac{d(z,Jz).d(Jz,STz)+d(STz,z).d(z,z)}{d(z,Jz)+d(STz,z)}\right]+\alpha_2[d(z,z)+d(Jz,STz)]$$
$$+\alpha_3[d(z,Jz)]+F(d(STz,z).d(z,z))$$

which implies that

$$d(z,STz) \le \alpha_3 d(z,STz)$$

STz = *z* = *Jz*, It follows from the upper part.
Since *ST*(X) $\subset$ *I*(*X*) there always exist a point z'' in X .such that $Iz'' = z$. Thus

$$d(ABz'',z) = d(ABz'',STz)$$

$$\le \alpha_1 \left[\frac{d(ABz'',Jz).d(Jz,STz) + d(STz,Iz'').d(Iz'',ABz'')}{d(ABz'',Jz) + d(STz,Iz'')} \right] + \alpha_2 [d(ABz'',Iz'') + d(Jz,STz)] + \alpha_3 d(Iz'',Jz) + F(d(STz,Iz'').d(Iz'',ABz''))$$

$$\le \alpha_1 \left[\frac{d(ABz'',z).d(z,z) + d(z,z).d(z,ABz'')}{d(ABz'',z) + d(z,z)} \right] + \alpha_2 [d(ABz'',z) + d(z,z)] + \alpha_3 d(z,z) + F(d(z,z).d(z,ABz'')) = \alpha_2 d(ABz'',z) + F(0)$$

equivalently,

$$d(ABz'',z) \le \alpha_2 d(ABz'',z)$$

which shows that $ABz'' = z$.
Also since (*AB*, *I*) are compatible and hence weakly commuting we obtain.

$$d(ABz,Iz) = d(AB(Iz''),I(ABz'')) \le d(Iz'',ABz'') = d(z,z) = 0$$

Therefore $ABz = Iz = z$.

Thus we have proved that *z* is a common fixed point of *AB*, *ST*, *I* and *J*.

Instead of *AB* or *I* if mappings *ST* or *J* is continuous, then the proof that *z* is a common fixed point of *AB*, *ST*, *I* and *J* is similar.

To show that *z* is unique, Let υ be the another fixed point of *I*, *J*, *AB* and *ST* then

$$d(z,\upsilon) = d(ABz,ST\upsilon)$$

$$\le \alpha_1 \left[\frac{d(ABz,J\upsilon).d(J\upsilon,ST\upsilon) + d(ST\upsilon,Tz).d(Iz,ABz)}{d(ABz,J\upsilon) + d(ST\upsilon,Iz)} \right] + \alpha_2 [d(ABz,Iz) + d(J\upsilon,ST\upsilon)] + \alpha_3 d(Iz,J\upsilon) + F(d(ST\upsilon,Iz).d(Iz,ABz))$$

or $$d(z,\upsilon) \le \alpha_3 d(z,\upsilon) + F(0)$$

or $$d(z,\upsilon) \le \alpha_3 d(z,\upsilon)$$

yielding there by $z = \upsilon$

Finally we need to show that *z* is also a common fixed point of *A*,*B*,*S*,*T*,*I* and *J*. For this let z be the unique common fixed point of both the pairs (*AB*, *I*) and (*ST*, *J*) then.

$$Az = A(ABz) = A(BAz) = AB(Az), \quad Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz),$$
$$Bz = B(Iz) = I(Bz)$$

which show that Az and Bz is a common fixed point of (AB, I) yielding thereby $Az=z=Bz=Iz=ABz$ in the view of uniqueness of common fixed point of the pair (AB, I).

Similarly using the commutativity of (S,T), (S,J) and (T,J) it can be shown that $Sz = z = Tz = Jz = STz$
Now we need to show that $Az=Sz$ ($Bz=Tz$) also remains a common fixed point of both the pairs (AB,I) and (ST, J) for this

$$d(Az,Sz) = d(A(BAz),S(TSz))$$
$$= d(AB(Az),ST(Sz))$$

$$\leq \alpha_1\left[\frac{d(AB(Az),J(Sz)).d(J(Sz),ST(Sz)) + d(ST(Sz),I(Az)).d(I(Az),AB(Az))}{d(AB(Az),J(Sz)) + d(ST(Sz),I(Az))}\right]$$
$$+\alpha_2[d(AB(Az),I(Az)) + d(J(Sz),ST(Sz))] + \alpha_3 d(I(Az),J(Sz))$$
$$+F(d(ST(Sz),I(Az)).d(I(Az),AB(Az)))$$

implies that $d(Az,Sz) = 0(as\ d(AB(Az)),J(Sz)) + (ST(Sz),I(Az)) = 0)$,
using condition (2), yielding thereby $Az = Sz$. Similarly it can be shown that $Bz = Tz$.

Thus z is the unique common fixed point of A,B,S,T,I, and J

Case – II :
Suppose that $d(ABx, Jy) + d(STy, Ix) = 0$ implies that $d(ABx, STy) = 0$, Then we argue as follows:
Suppose that there exists an n such that $z_n = z_{n+1}$
Then, also $z_{n+1} = z_{n+2}$, suppose not. Then from (3) we have
$0 < d(z_{n+1}, z_{n+2}) \leq kd(z_{n+1}, z_n)$ yielding there $z_{n+1} = z_{n+2}$. Thus $z_n=z_{n+k}$ for k = 1,2,... It then follows that there exist two point w_1 and w_2 such that $\upsilon_1 = ABw_1 = Jw_2$ and $\upsilon_2 = STw_2 = Iw_1$.
Since $d(ABw_1, Jw_2) + d(STw_2, Iw_1) = 0$, from (3)
$d(ABw_1STw_2) = 0 \quad i.e \quad \upsilon_1 = ABw_2 = STw_2 = v_2$
Note also that $Iv_1 = I(ABw_1) = AB(Iw_1) = ABv_2$
Similarly $ST\upsilon_2 = J\upsilon_2$. Define $y_1 = AB\upsilon_1, y_2 = ST\upsilon_2$,
since $d(AB\upsilon_1, J\upsilon_2) + d(ST\upsilon_2, J\upsilon_1) = 0$
it follows from (2.3.9) that $d(AB\upsilon_1, ST\upsilon_2) = 0$ i.e. $y_1 = y_2$
Thus $AB\upsilon_1 = I\upsilon_1 = ST\upsilon_2 J\upsilon_2$ But $\upsilon_1 = \upsilon_2$

Therefore AB,I,ST and J have a common coincidence points define $w = AB\upsilon_1$, it then follows that w is also a common coincidence point of AB,I, ST and J, if $ABw \neq AB\upsilon_1 = ST\upsilon_1$ then d $(ABw, ST\upsilon_1) > 0$, But, since $d(ABw, J\upsilon) + d(ST\upsilon_1, Iw) = 0$. if follows from (2) that $d(ABw, ST\upsilon_1) = 0,$ i.e. $ABw = ST\upsilon_1$, a contradiction.

Therefore $ABw = ST\upsilon_1 = w$ and w is a common fixed point of AB, ST, I and J.

The other part is identical to the case (1), hence it is omitted, this complete the proof.

If $F(t) = 0$, for all $t \in R^+$ and putting $\alpha_2 = 0$, $AB = A$, $ST = B$, this will give the following result of Jeong-Rhoades [6]

2.3 Corollary :
Let A,B,S and T be for self maps of a complete metric space (X,d) satisfying $A(X) \subset T(X), B(X), \subset S(X),$ and for each $x, y \in X$ either.

$$d(Ax,By)$$
$$\leq \alpha\left[\frac{d(Ax,Sx).d(Sx,By) + d(By,Ty).d(Ty,Ax)}{d(Sx,By) + d(Ty,Ax)}\right] + \beta d(Sx,Ty)$$

if $d(Sx, By) + d(Ty, Ax) \neq 0, \alpha, \beta \geq 0$, $\alpha + \beta < 1$ or
$d(Ax, By) = 0$ *if* $d(Sx, By) + d(Ty, Ax) = 0$
if either
(a) (*A*, *S*) are compatible, *A* or *S* is continuous and (*B*,*T*) are weakly compatible or
(a') (*B*,*T*) are compatible, *B* or *T* is continuous and (*A*,*S*) are weakly compatible, then *A*,*B*,*S* and *T* have a unique common fixed point *z*. Moreoverz is the unique common fixed point of *A* and *S* and of *B* and *T*.

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