International Journal of Psychosocial Rehabilitation, Vol. 23, Issue 1, 2019
ISSN: 1475-7192

Common Fixed Point Theorems For Weakly Compatible Mappings Satisfying Rational Contractive Conditions In Complete Metric Spaces

[1]*A.K. Goyal

Abstract
By using notions of compatibility, weak compatibility and commutativity, Goyal ([5], [6]) prove some common fixed point theorems for six mappings involving rational contractive conditions in complete metric spaces. In this paper , we prove a common fixed point theorem for three pairs of weakly compatible mappings in complete metric spaces satisfying a rational inequality without any continuity requirement which generalize several previously known results due to Imdad and Ali [12], Goyal [5], Imdad-Khan [13], Jeong-Rhoades [7] and others.

Keywords: *Complete metric spaces, fixed points, compatible mapping, weak compatible mapping*.

2000 AMS SUBJECT CLASSIFICATION CODE: 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES:

In recent years several definitions of conditions weaker than commutativity have appeared which facilitated significantly to extend the Jungck's theorem and several others. Foremost among them is perhaps the weak commutativity condition introduced by Sessa [17] which can be described as follows:

1.1 Definition:
Let S and T be mappings of a metric space (X, d) into itself. Then (S, T) is said to be **weakly commuting** pair if
$d(STx, TSx) \le d(Tx, Sx)$ for all $x \in X$.
obviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

1.2 Example:

Consider the set $X = [0, 1]$ with the usual metric. Let $Sx = \frac{x}{2}$ and $Tx = \frac{x}{2+x}$ for every $x \in X$. Then for all $x \in X$

$$STx = \frac{x}{4+2x}, \quad TSx = \frac{x}{4+x}$$

hence $ST \neq TS$. Thus$S$ and T do not commute.
Again

$$d(STx, TSx) = \left| \frac{x}{4+2x} - \frac{x}{4+x} \right| = \frac{x^2}{(4+x)(4+2x)}$$

$$\le \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(Sx, Tx)$$

and so S and T commute weakly.
Obviously, the class of weakly commuting is wider and includes commuting mappings as subclass.
Jungck [9] has observed that for $X = R$ if $Sx = x^3$ and $Tx = 2x^3$ then S and T are not weakly commuting. Thus it is desirable to a less restrictive concept which he termed as 'compatibility' the class of compatible mappings is still wider and includes weakly commuting mappings as subclass as is evident from the following definition of Jungek [9.

[1]* Department of Mathematics, M. S. J. Govt. P.G. College, Bharatpur (Raj.)-321001, Email: akgbpr67@rediffmail.com

1.3 Definition:

Two self mappings$S$ and T of a metric space (X, d) are **compatible** if and only if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X. such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly any weakly commuting pair $\{S,T\}$ in compatible but the converse need not be true as can be seen in the following example.

1.4 Example:

Let $Sx = x^3$ and $Tx = 2x^3$ with $X = R$ with the usual metric. Then S and T are compatible, since

$|Tx - Sx| = |x^3| \to 0$ if and only if

$|STx - TSx| = 6|x^9| \to 0$ but

$|STx - TSx| \le |Tx - Sx|$ is not true for all $x \in X$, say for example at $x = 1$.

1.5 Proposition:

Let S and T be continuous self mapping on X. Then the pair (S, T) is compatible on X. where as in (Jungck [11], Gajic [4]) demonstrated by suitable examples that if S and T are discontinuous then the two concepts are independent of each other. The following examples also support this observation.

1.6 Example:

Let $X = R$ with the usual metric we define $S, T: X \longrightarrow X$ as follows.

$$Sx = \begin{cases} 1/x^2 & x \ne 0 \\ 0 & x = 0 \end{cases} \quad and \quad Tx = \begin{cases} 1/x^3 & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Both S and T are discontinuous at $x = 0$ and for any sequence $\{x_n\}$ in X, we have $d(STx_n, TSx_n) = 0$. Hence the pair (S,T) is compatible.

1.7 Example:

Now we define

$$Sx = \begin{cases} 1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases} \quad and \quad Tx = \begin{cases} -1/x^3, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

observe that the restriction of S and T on $(-\infty, 1]$ are equal.

Thus we take a sequence $\{x_n\}$ in $(1, \infty)$. Then $\{Sx_n\} \subset (0,1)$ and $\{Tx_n\} \subset (-1,0)$. Thus for every n, $TTx_n = 0$, $TSx_n = 1$, $STx_n = 0$, $SSx_n = 1$. So that $d(STx_n, TTx_n) = 0$, $d(TSx_n, TTx_n) = 0$ for every $n \in N$. This shows that the pair (S, T) is compatible of type (A). Now let $x_n = n$, $n \in N$. Then $Tx_n \to 0,\ Sx_n \to 0$ as $n \to \infty$ and $STx_n = 0,\ TSx_n = 1$ for every $n \in N$ and so

$d(STx_n, TSx_n) \ne 0$ as $n \to \infty$ hence the pair (S, T) is not compatible.

Very recently concept of **weakly compatible** obtained by Jungck-Rhoades [8] stated as the pair of mappings is said to be weakly compatible if they commute at their coincidence point.

1.8 Example:

Let X = [2, 20] with usual metric define

$$Tx = \begin{cases} 2 & if \ \ x = 2 \\ 12 + x & if \ \ 2 < x \le 5 \\ x - 3 & if \ \ 5 < x \le 20 \end{cases} \quad and \quad Sx = \begin{cases} 2 & if \ \ x \in \{2\} \cup (5,20] \\ 8 & if \ \ 2 < x \le 5 \end{cases}$$

S and T are weakly compatible mappings which is not compatible.

2. MAIN RESULTS:

Let R^+ be the set of non-negative real numbers and let $F:R^+ \longrightarrow R^+$ be a mapping such that $F(0)$ and F is continuous at 0. The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [9].

2.1 Lemma:

Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists a $k \in (0,1)$ such that $d(y_{n+1}, y_n) \leq k\ (y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Motivated by the contractive condition given by, Jeong Rhoades [7] and Nesic [16]we prove the following theorem.

Theorem 2.1: Let (X, d) be a complete metric space. Let A, B, S, T, I and J be self-mappings of a complete metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$ either

$$d(ABx, STy) \leq \beta_1 \left[\frac{\{d(ABx, Ix)\}^2 + \{d(STy, Jy)\}^2}{d(ABx, Ix) + d(STy, Jy)}\right] + \beta_2 d(Ix, Jy)$$
$$+\beta_3[d(ABx, Jy) + d(STy, Ix)]$$
$$+F(min\{d^2(Ix, Jy), d(Ix, ABx).d(Ix, STy), d(Jy, STy).d(Jy, ABx)\}) \quad \ldots (1)$$

if $d(ABx, Ix) + d(STy, Jy) \neq 0$, $\beta_i \geq 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1 + \beta_2 + 2\beta_3 < 1$

or, $\quad d(ABx, STy) = 0$ if $d(ABx, Ix) + d(STy, Jy) = 0 \quad \ldots (2)$

If one of the $AB(X), ST(X), J(X)$ and $I(X)$ is a complete subspace of X, then

(a) (AB, I) has a coincidence point

(b) (ST, J) has a coincidence point

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point. Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. Since $AB(X) \subset J(X)$, we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Again, since $ST(X) \subset I(X)$, we can choose a point x_2 in X with $STx_1 = Ix_2$. Using this process repeatedly, we can construct a sequence $\{z_n\}$ such that

$z_{2n} = ABx_{2n} = Jx_{2n+1}$ and $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$

Now, we consider two cases

Case I: If $d(ABx, Ix) + d(STy, Jy) \neq 0$. Then on using inequality (1), we have

$$d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$
$$\leq \beta_1 \left[\frac{\{d(ABx_{2n+2}, Ix_{2n+2})\}^2 + \{d(STx_{2n+1}, Jx_{2n+1})\}^2}{d(ABx_{2n+2}, Ix_{2n+2}) + (STx_{2n+1}, Jx_{2n+1})}\right] + \beta_2 d(Ix_{2n+2}, Jx_{2n+1})$$
$$+\beta_3[(d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})]$$
$$+F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}),$$
$$d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}),$$
$$d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}]$$
$$\leq \beta_1 \frac{[d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})]^2}{d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})} + \beta_2 d(Ix_{2n+2}, Jx_{2n+1})$$
$$+\beta_3[d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})]$$
$$+F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}),$$
$$d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}),$$
$$d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}]$$
$$\leq \beta_1[d(ABx_{2n+2}, Ix_{2n+2}) + d(STx_{2n+1}, Jx_{2n+1})]$$
$$+\beta_2 d(Ix_{2n+2}, Jx_{2n+1})$$
$$+\beta_3[d(ABx_{2n+2}, Jx_{2n+1}) + d(STx_{2n+1}, Ix_{2n+2})]$$
$$+F[min\{d^2(Ix_{2n+2}, Jx_{2n+1}),$$
$$d(Ix_{2n+2}, ABx_{2n+2}).d(Ix_{2n+2}, STx_{2n+1}),$$
$$d(Jx_{2n+1}, STx_{2n+1}).d(Jx_{2n+1}, ABx_{2n+2})\}]$$
$$\leq \beta_1[d(z_{2n+2}, z_{2n+1}) + d(z_{2n+1}, z_{2n})] + \beta_2 d(z_{2n+1}, z_{2n})$$
$$+\beta_3[d(z_{2n+2}, z_{2n+1}) + d(z_{2n+1}, z_{2n})]$$
$$+F[min\{d^2(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}).d(z_{2n+1}, z_{2n+1}),$$
$$d(z_{2n}, z_{2n+1}).d(z_{2n}, z_{2n+2})\}]$$
$$\leq (\beta_1 + \beta_3)\, d(z_{2n+2}, z_{2n+1}) + (\beta_1 + \beta_2 + \beta_3) d(z_{2n}, z_{2n+1})$$
$$+F[min\{d^2(z_{2n+1}, z_{2n}),$$
$$d(z_{2n+1}, z_{2n+2}).0, d(z_{2n}, z_{2n+1}).\, d(z_{2n}, z_{2n+2})\}]$$

or, $d(z_{2n+2}, z_{2n+1}) \leq \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n+1}, z_{2n})$
$+\frac{1}{(1-\beta_1-\beta_3)} F[min\{d^2(z_{2n+1}, z_{2n}),$
$0, d(z_{2n}, z_{2n+1}). d(z_{2n}, z_{2n+2})\}]$
$= \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n+1}, z_{2n}) + \frac{1}{(1-\beta_1-\beta_3)} F(0)$
or, $d(z_{2n+1}, z_{2n+2}) \leq \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n}, z_{2n+1}) + 0$ $[\because F(0) = 0]$
or, $d(z_{2n+1}, z_{2n+2}) \leq \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n}, z_{2n+1})$
Following the same process, we can show that
$$d(z_{2n}, z_{2n+1}) \leq \left(\frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3}\right) d(z_{2n-1}, z_{2n})$$
Thus, for every n, we can show that
$$d(z_n, z_{n+1}) \leq \alpha d(z_{n-1}, z_n) \quad \ldots (3)$$
Where $\alpha = \frac{\beta_1+\beta_2+\beta_3}{1-\beta_1-\beta_3} < 1$
Now, by induction
$$\begin{aligned} d(z_n, z_{n+1}) &\leq \alpha d(z_{n-1}, z_n) \\ &\leq \alpha^2 d(z_{n-2}, z_{n-1}) \\ &\vdots \\ &\leq \alpha^n d(z_0, z_1) \end{aligned}$$
For any $m > n$, we get,
$$\begin{aligned} d(z_n, z_m) &\leq d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \ldots + d(z_{m-1}, z_m) \\ &\leq [\alpha^n + \alpha^{n+1} + \ldots\ldots\ldots + \alpha^{m-1}] d(z_0, z_1) \\ &\leq \frac{\alpha^n}{1-\alpha} d(z_0, z_1) \end{aligned}$$
This implies that $d(z_n, z_m) \to 0$ as $n, m \to \infty$
Hence, sequence $\{z_n\}$ described by
$$\{ABx_0, STx_1, ABx_2, \ldots STx_{2n-1}, ABx_{2n}, STx_{2n+1}, \ldots\}$$
is a Cauchy sequence in a complete metric space (X, d). Now, let $ST(X)$ is a complete subspace of X, then the subsequence $\{z_{2n+1}\}$ which is contained in $ST(X)$ also get a limit z in $ST(X)$ i.e.
$$\lim_{n\to\infty} ST\, x_{2n+1} = z$$

Since, $ST(X) \subset I(X)$, there exists a point $z' \in X$ such that $Iz' = z$.
Again, as $\{z_n\}$ is a Cauchy sequence containing a convergent subsequence $\{z_{2n+1}\}$, therefore the sequence $\{z_n\}$ also converges which implies the convergence of $\{z_{2n}\}$ being a subsequence of the convergent sequence $\{z_n\}$ i.e. $\lim_{n\to\infty} J\, x_{2n+1} z$.
To prove that $ABz' = z$ put $x = z'$ and $y = x_{2n-1}$ in (1), we get
$$\begin{aligned} d(ABz', STx_{2n-1}) \leq \beta_1 &\left[\frac{\{d(ABz', Iz')\}^2 + \{d(STx_{2n-1}, Jx_{2n-1})\}^2}{d(ABz', Iz) + d(STx_{2n-1}, Jx_{2n-1})}\right] + \beta_2 d(Iz', Jx_{2n-1}) \\ &+\beta_3[d(ABz', Jx_{2n-1}) + d(STx_{2n-1}, Iz')] \\ &+F[min\{d^2(Iz', Jx_{2n-1}), \\ &\quad d(Iz', ABz'). d(Iz', STx_{2n-1}), \\ &\quad d(Jx_{2n-1}, STx_{2n-1}). d(Jx_{2n-1}, ABz')\}] \end{aligned}$$
on letting $n \to \infty$, above reduces to
$$\begin{aligned} d(ABz', z) &\leq \beta_1 \left[\frac{\{d(ABz', z)\}^2 + \{d(z, z)\}^2}{d(ABz', z) + d(z, z)}\right] + \beta_2 d(z, z) + \beta_3[d(ABz', z) + d(z, z)] \\ &\quad +F[min\{d^2(z, z), d(z, ABz'). d(z, z), d(z, z). d(z, ABz')\}] \\ &\leq \beta_1 d(ABz', z) + \beta_3 d(ABz', z) \\ &\quad +F[min\{0, d(z, ABz'). 0, 0. d(z, ABz')\}] \\ &\leq (\beta_1 + \beta_3) d(ABz', z) + F(0) \end{aligned}$$
or, $d(ABz', z) \leq (\beta_1 + \beta_3)\, d(ABz', z)$ $[\because F(0) = 0]$

which gives $ABz' = z$ [by using Remark (1.16)].
Thus, we get $ABz' = Iz' = z$ and result (a) is established i.e the pair (AB, I) has a coincidence point.
Since z is in the range of AB i.e. $ABz' = z$ and $AB(X) \subset J(X)$ there always exists a point z" such that $Jz" = z$
Now, $d(z, STz") = d(ABz', STz")$
$$\begin{aligned} &\leq \beta_1 \left[\frac{\{d(ABz', Iz')\}^2 + \{d(STz, Jz)\}^2}{d(ABz', Iz') + d(STz", Jz")}\right] + \beta_2 d(Iz', Jz") \\ &\quad +\beta_3 d(ABz', Jz") + d(STz", Iz') \\ &\quad +F[min\{d^2(Iz', Jz"), d(Iz', ABz'). d(Iz', STz"), \\ &\qquad d(Jz", STz"). d(Jz", ABz')\}] \end{aligned}$$

$$\leq \beta_1 \left[\frac{\{d(z,z)\}^2+\{d(STz",z)\}^2}{d(z,z)+d(STz",z)}\right] + \beta_2 d(z,z) + \beta_3[d(z,z) + d(STz",z)]$$
$$+F[min\{d^2(z,z), d(z,z).d(z,STz"), d(z,STz").d(z,z)\}]$$
$$\leq (\beta_1+\beta_3)\ d(z,STz") + F[min\{0,0,0\}]$$

or, $d(z,STz") \leq (\beta_1+\beta_3)d(z,STz") + F(0)$

or, $d(z,STz") \leq (\beta_1+\beta_3) + d(z,STz")[\because F(0)=0]$

which implies that $STz" = z = Jz"$ i.e. the pair (ST,J) has a coincidence point. This establishes the result (b).

If we assume that $I(X)$ is a complete subspace of X, then similar arguments establish results (a) and (b). The remaining two cases pertain essentially to the previous cases.

Infact, if $ST(X)$ is complete then $z \in ST(X) \subset I(X)$ and if $AB(X)$ is complete, then, $z \in AB(X) \subset J(X)$.

Thus, the results (a) and (b) are completely established.

Furthermore, if the pairs (AB,I) and (ST,J) are coincidentally commuting at z' and $z"$ respectively then

(i) $z = ABz' = Iz' = STz" = Jz"$

(ii) $ABz = AB(Iz') = I(ABz') = Iz$

(iii) $STz = ST(Jz") = J(STz") = Jz$

Since, $d(ABz',Iz') + d(STz,Jz) = 0$

Therefore, by (2), we get $d(ABz',STz) = d(z,STz) = 0$

or, $z = STz$.

Similarly, $d(ABz,Iz) + d(STz",Jz") = 0$, therefore by (2), we get

$d(ABz,STz") = d(ABz,z) = 0$

or, $z = ABz$.

Thus, $ABz = Iz = STz = Jz = z$, which shows that z is a common fixed point of AB, ST, I and J.

To show that z is unique, let u be another fixed point of I, J, AB and ST. Then,

$$d(z,u) = d(ABz,STu)$$
$$\leq \beta_1 \left[\frac{\{d(ABz,Iz)\}^2+\{d(STu,Ju)\}^2}{d(ABz,Iz)+d(STu,Ju)}\right] + \beta_2 d(Iz,Ju)$$
$$+\beta_3[d(ABz,Ju) + d(STu,Iz)]$$
$$+F[min\{d^2(Iz,Ju), d(Iz,ABz).d(Iz,STu),$$
$$d(Ju,STu).d(Ju,ABz)\}]$$
$$\leq \beta_1 \left[\frac{\{d(ABz,Iz)+d(STu,Ju)\}^2}{d(ABz,Iz)+d(STu,Ju)}\right] + \beta_2 d(Iz,Ju)$$
$$+\beta_3[d(ABz,Ju) + d(STu,Iz)]\ +F[min\{d^2(Iz,Ju), d(Iz,ABz).d(Iz,STu),$$
$$d(Ju,STu).d(Ju,ABz)\}]$$
$$\leq \beta_1[d(ABz,Iz) + d(STu,Ju)] + \beta_2 d(Iz,Ju)$$
$$+\beta_3[d(ABz,Ju) + d(STu,Iz)]$$
$$+F[min\{d^2(Iz,Ju), d(Iz,ABz).d(Iz,STu),$$
$$d(Ju,STu).d(Ju,ABz)\}]$$
$$\leq (\beta_2+2\beta_3)d(z,u)$$
$$+F[min\{d^2(z,u), d(z,z).d(z,u), d(u,u).d(u,z)\}]$$
$$\leq (\beta_2+2\beta_3)d(z,u) + F[min\{d^2(z,u),0,0\}]$$
$$\leq (\beta_2+2\beta_3)d(z,u) + F(0)$$
$$\leq (\beta_2+2\beta_3)d(z,u) \qquad [\because F(0)=0]$$

yielding, thereby $z = u$.

Thus, z is a unique common fixed point of AB, ST, I and J.

Finally, we prove that z is also a common fixed point A,B, S,T, I and J. For this, let both the pairs (AB, I) and (ST, J) have a unique common fixed point z.

Then $Az = A(ABz) = A(BAz) = AB(Az)$

$Az = A(Iz) = I(Az)$

$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz)$

$Bz = B(Iz) = I(Bz)$

which shows that (AB, I) has common fixed points, which are Az and Bz. We get thereby, $Az = z = Bz = Iz = ABz$, by virtue of uniqueness of common fixed point of pair (AB, I).

Similarly, using the commutativity of (S,T), (S,J) and (T,J),

$Sz = z = Tz = Jz = STz$ can be shown.

Now, to show that Az = Sz (Bz = Tz), we have

$$d(Az,Sz) = d\big(A(BAz),S(TSz)\big) = d(AB(Az),ST(Sz))$$
$$\leq \beta_1 \left[\frac{\{d(AB(Az),I(Az))\}^2+\{d(ST(Sz),J(Sz))\}^2}{d(AB(Az),I(Az))+d(ST(Sz),J(Sz))}\right]$$

$$+\beta_2 d\big(I(Az), J(Sz)\big)$$
$$+\beta_3\big[d\big(AB(Az), J(Sz)\big) + d\big(I(Az), ST(Sz)\big)\big]$$
$$+F\big[min\big\{d^2\big(I(Az), J(Sz)\big),$$
$$d\big(I(Az), AB(Az)\big).\, d\big(I(Az), ST(Sz)\big),$$
$$d\big(J(Sz), ST(Sz)\big).\, d\big(J(Sz), AB(Az)\big)\big\}\big]$$

which implies that $d(Az, Sz) = 0$
(as $d\big(AB(Az), I(Az)\big) + d\big(ST(Sz), J(Sz)\big) = 0$), using condition (2), thereby we get $Az = Sz$.
Similarly, $Bz = Tz$can be shown.
Hence, z is a unique common fixed point of A, B, S, T, I and J.

Case II: Let $d(ABx, Ix) + d(STy, Jy) = 0$ implies that $d(ABx, STy) = 0$. Then we argue as follows:
Here we show that if $y_n = y_{n+1}$ for some n, then AB, ST, I and J have a common fixed point.
Suppose that there exists as n such that $z_n = z_{n+1}$. Then also $z_{n+1} = z_{n+2}$.
For if, $z_{n+1} \neq z_{n+2}$, then from (3), with n replaced by $n + 1$, we get,

$0 < d(z_{n+1}, z_{n+2}) = 0$ a contradiction, gives $z_{n+1} = z_{n+2}$.

Thus, $z_n = z_{n+\alpha}$ for $\alpha = 1,2, \ldots$
If follows that there exists two points u_1 and u_2 such that $v_1 = ABu_1 = Iu_1$ and $v_2 = STu_2 = Ju_2$. Since $d(ABu_1, Iu_1) + d(STu_2, Ju_2) = 0$ then from (2), we get
$d(ABu_1, STu_2) = 0$ i.e $v_1 = ABu_1 = STu_2 = v_2$
Also, note that $Iv_1 = I(ABu_1) = AB(Iu_1) = ABv_1$.
Similarly, $Jv_2 = J(STu_2) = ST(Ju_2) = STv_2$.
Define$y_1 = ABv_1$, $y_2 = STv_2$
Since $d(ABv_1, Iv_1) + d(STv_2, Jv_2) = 0$ it follows from (2) that

$d(ABv_1, STv_2) = 0$

or, $ABv_1 = STv_2$ i.e $y_1 = y_2$.
Thus $ABv_1 = Iv_1 = STv_2 = Jv_2$
But, $v_1 = v_2$, therefore AB, I, ST and J have a common coincidence point.
Define $u = ABv_1$, which asserts that u is also a common point of coincidence of AB, ST, I and J. If $ABu \neq ABv_1 = STv_1$, then $d(ABu, STv_1) > 0$ but since $d(ABu, Iu) + d(STv_1, Jv_1) = 0$, it follows from (2) that $(ABu, STv_1) = 0$, i.e $ABu = STv_1$ which is a contradiction. Therefore, $ABu = ABv_1 = u$ and u is a common fixed point of AB, ST, I and J.
The rest of the proof is identical to the case(I), hence it is omitted.
This completes the proof.
If we put $F(t) = 0$ for all $t \in R^+$ in theorem (2.1), we obtain the following, which generalize the result of Imdad and Ali [12] in complete metric space for six mappings.

Corollary 2.2. Let (X, d) be a complete metric space. Let A, B, S, T, I and J be self-mappings of a complete metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x, y \in X$ either

$$d(ABx, STy) \leq \beta_1\left[\frac{\{d(ABx, Ix)\}^2 + \{d(STy, Jy)\}^2}{d(ABx, Ix) + d(STy, Jy)}\right] + \beta_2 d(Ix, Jy)$$
$$+\beta_3[d(ABx, Jy) + d(STy, Ix)]$$

if $d(ABx, Ix) + d(STy, Jy) \neq 0$, $\beta_i \geq 0$ $(i = 1,2,3)$ with at least one β_i non zero and $2\beta_1 + \beta_2 + 2\beta_3 < 1$
or, $d(ABx, STy) = 0$ if $d(ABx, Ix) + d(STy, Jy) = 0$
If one of the $AB(X), ST(X), J(X)$ and $I(X)$ is a complete subspace of X, then
(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point
Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point. Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.
Putting $AB = A, ST = B$ in corollary (2.2), we obtain the following generalization of the result of Imdad and Ali [12] in complete metric space.

Corollary 2.3: Let (X, d) be a complete metric space.Let A, B, S and T be self-mappings of a complete metric space (X, d) with $A(X) \subset T(X)$and $B(X) \subset S(X)$ such that for each $x, y \in X$ either

$$d(Ax, By) \leq \beta_1\left[\frac{\{d(Ax, Sx)\}^2 + \{d(By, Ty\}^2}{d(Ax, Sx) + d(By, Ty)}\right] + \beta_2\, d(Sx, Ty)$$
$$+\beta_3[d(Ax, Ty) + d(Bx, Sx)]$$

If $d(Ax, Sx) + d(By, Ty) \neq 0, \beta_i \geq 0$ $(i = 1,2,3)$ (with at least one β_i non zero) and $2\beta_1 + \beta_2 + 2\beta_3 < 1$or $d(Ax, By) = 0$ whereever
$d(Ax, Sx) + d(By, Ty) = 0$.
If one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$is a complete subspace of X, then

(a) (A,S) has a coincidence point
(b) (B,T) has a coincidence point
Further, if the pairs (A,S) and (B,T) are coincidentally commuting then A, B, S and T has a unique fixed point z.
On the basis of the above corollary (2.2), we have the following result, whose proof is similar to that of corollary (2.2).

Corollary 2.4: Let (X,d) be a complete metric space. Let A, B, S, T, I and J be self-mappings of a complete metric space (X,d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ such that for each $x,y \in X$.

$$d(ABx,STy) \leq \beta_1[d(ABx,Ix)+d(STy,Jy)]+\beta_2 d(Ix,Jy) +\beta_3[d(ABx,Jy)+d(STy,Ix)]$$

where $\beta_i \geq 0, (i=1,2,3)$ (with at least one β_i non zero) and $2\beta_1+\beta_2+2\beta_3 < 1$
If one of the $AB(X), ST(X), J(X)$ and $I(X)$ is a complete subspace of X, then
(a) (AB, I) has a coincidence point
(b) (ST, J) has a coincidence point
Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point.
Moreover, if the pairs (A,B), (A,I), (B,I), (S,T), (S,J) and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof: Since

$$\frac{[d(ABx,Ix)]^2+[d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,Gy)} \leq \frac{[d(ABx,Ix)+d(STy,Jy)]^2}{d(Ax,Fx)+d(Sy,Gy)} = d(ABx,Ix)+d(STy,Jy)$$

Using above inequality in main Theorem (2.1), we get the corollary (2.4).
Taking $AB=A, ST=B, I=J=S$ in corollary (2.4), we obtain the following result.

Corollary 2.5: Let (X,d) be a complete metric space. Let A, B and S be self-mappings of a complete metric space (X,d) satisfying $A(X) \subset S(X)$, $B(X) \subset S(X)$ such that for each $x,y \in X$.

$$d(Ax,By) \leq \beta_1[d(Ax,Sx)+d(By,Sy)]+\beta_2 d(Sx,Sy) +\beta_3[d(Ax,Sy)+d(By,Sx)]$$

where $\beta_i \geq 0, (i=1,2,3)$ (with at least one β_i non zero) and $2\beta_1+\beta_2+2\beta_3 < 1$
If one of the $A(X), B(X)$ and $S(X)$ is a complete subspace of X, then the pair (AB,S) have unique coincidence point.
Further, if the pairs (A, S) and (B, S) are coincidentally commuting (weakly compatible), then A, B and S have a unique common fixed point.
Now, we furnish an example to demonstrate the validity of the hypothesis of our Corollary(2.2).

Example 2.6: Consider $X=[0,1]$ with the usual metric defined by

$d(x,y)=\|x-y\|=|x-y|$ and $F=R=$ Real Banach space.

Define self mappings A, B, S, T, I and J on X by

$$Ax=\frac{3x}{8}, Bx=\frac{4x}{10}, Sx=\frac{x}{5}, Tx=\frac{5x}{12}, Ix=\frac{3x}{20}, Jx=\frac{x}{3}$$

Here, $ABx=A\left(\frac{4x}{10}\right)=\frac{3}{8}\left(\frac{4x}{10}\right)=\frac{3}{20}x$

$STx=S\left(\frac{5x}{12}\right)=\frac{1}{5}\left(\frac{5x}{12}\right)=\frac{x}{12}$

$\therefore$ $AB(X)=\left[0,\frac{3}{20}\right] \subset \left[0,\frac{1}{3}\right]=J(X)$

$ST(X)=\left[0,\frac{1}{12}\right] \subset \left[0,\frac{3}{20}\right]=I(X)$

or, $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$
Here all the contractive condition of the Corollary (2.2) are satisfied. Hence, mappings A, B, S, T, I and J have a unique common fixed point at $x=0$.
Now, we furnish an example to demonstrate the validity of the hypothesis of our corollary (2.3).

Example 2.7: Consider $X=[0,8]$ with the usual metric defined by

$d(x,y)=\|x-y\|=|x-y|$ and $F=R=$ Real Banach space.

Define self mappings A, B, S and T on X as

$A0=0,\ Ax=1,\ 0<x\leq 8$
$B0=0,\ Bx=1,\ 0<x<8,\ B8=0$
$S0=0,\ Sx=7,\ 0<x<8,\ S8=4$
$T0=0,\ Tx=8,\ 0<x<8,\ T8=1$

Here all the four maps in this example are discontinuous even at their unique common fixed point 0.
Here, $A(X)=\{0,1\} \subset T(X)=\{0,1,8\}$
And $B(X)=\{0,4\} \subset S(X)=\{0,4,7\}$

Also, the pair (A, S) and (B, T) are coincidentally commuting at $x = 0$ which is their common coincidence point.
i.e. $A0 = S0 \Rightarrow AS0 = SA0$
$B0 = T0 \Rightarrow BT0 = TB0$
By a routine calculation, we can verify that all the contractive conditions of corollary (2.3) are satisfied for $\beta_1 = \frac{1}{20}, \beta_2 = \frac{1}{10}$ and $\beta_3 = \frac{3}{8}$. $(2\beta_1 + \beta_2 + 2\beta_3 = 0.95 < 1)$.

REFERENCES

[1] B. Fisher., "Mappings satisfying on fixed points", Bull. Cal. Math. Soc., ***60 (1968)***, 71-76.
[2] B. Fisher, "Common fixed point and constant mapping satisfying a rational inequality", Math. Sem. Notes., ***6 (1978)***, 29-35.
[3] B. Fisher., "Mapping satisfying rational inequality", Nanta. Math., ***12 (1979)***, 195-199.
[4] Lj. Gajic., "On common fixed point of compatible mappings of type (A) on metric and 2-metric spaces", Filomat (Nis)., ***10 (1996)***, 177-186.
[5] A. K. Goyal, "Common fixed point theorem for six mappings in complete metric spaces", Bull. Pure Appl. Math., 3 (1), 24–35, 2009.
[6] A. K. Goyal., "Common fixed point theorems for weakly compatible mappings satisfying rational contractive conditions", International Journal of Psychosocial Rehabilitation, Vol. 17(1),138-145, 2013
[7] G.S. Jeong and B.E. Rhoades., "Some remarks for improving fixed point theorem for more than two maps", Ind. J. Pure. Appl. Math ***28 (9) (1997)***, 1177-1196.
[8] G. Jungck and B.E. Rhoades., "Fixed point for set valued function without continuity", Ind. J. Pure. Appl. Math., ***29 (3) (1998)***, 227-238.
[9] J. Jungck., "Compatible mappings and common fixed point", Internet. J. Math and Math. Sci., ***9 (4) (1986)***, 771-779.
[10] G. Jungck., "Commuting mappings and fixed points", Amer. Math. Monthly., ***83 (1976)***, 261-263.
[11] J. Jungck, P.P. Murthy and Y.J. Cho., "Compatible mapping of type (A) and common fixed point theorems", Math. Japon., ***38 (2) (1993)***, 381-390.
[12] Imdad, M. and Ali, J., "Pairwise coincidentally commuting mappings satisfying a rational inequality", Italian Journal of Pure and Applied Mathematics, (20), 87-96, 2006.
[13] Imdad, M. and Khan, Q.H., "Six mappings satisfying a rational inequality", Radovi Matematicki, 9, 251–260, 1999.
[14] R. Kannan., "Some results on fixed point", Bull. Cal. Math. Soc., ***60 (1968)***, 71-76.
[15] R. Kannan., "Some results in fixed points", Amer. Math. Monthly., ***76 (1969)***, 405-408.
[16] Slobodan, C. Nesic., "Results on fixed points of aymptotically regular mappings", Ind. J. Pure. Appl. Math., ***30 (5) (1999),*** 491-494.
[17] S. Sessa., "On a weak commutativity condition in fixed point consideration", Publ. Inset. Math., ***32 (1982)***, 149-153.