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Laplace Transform: Developing the Variational Iteration Method

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Abstract:
The classification of the Lagrange multiplier plays an import rule in the variationaliteration method and the variational theory is widely used for this purpose. This paper suggests an easier approach by the Laplace transform to determining themultiplier, making the processobtainable to researchers facing different nonlinearproblems. A nonlinear oscillator is adopted as an illustration to elucidate thedetection process and the solution process, only one iteration leads to an ultimateresult.
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Introduction

The variational iteration method was proposed in late 1990s to solve aescape flow with fractionalderivatives and a nonlinear oscillator [1, 2], and this method has widely used as a main mathematicaltool to solving various nonlinear equations. Due to general study of the method by numerous authors, forexamples, Ji-Huan He [3–5], D.D. Ganji [6], T. Ozis and A. Yildirim [7], M.A. Noor and S.T. Mohyud-Din [8],it has completely developed into a fully fledged method in mathematics. Using "variational iteration method"as a searching topic in Clarivate's web of science, we found 3761 hits on 24 November 2018. The identificationof the Lagrange multiplier in the method requires the facts of the variational theory [8–11], and the complex detection process might delay applications of the method to practical problems. This paper suggestsan easier detection process by the Laplace transform, which is available in all mathematics textbooks.

The identification of the Lagrange multiplier by the Laplace transform
Consider a general non-linear oscillator equation in the form:

$$u''(t) + f(u) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with initial conditions $u(0) = A, \quad u'(0) = 0$ ……………………………………………………………………..(2)

We can rewrite Eq. (1)as

$$u'' + \omega^2 u + g(u) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

whereω is the frequency to be auxiliary determined, $g(u) = f(u) - \omega^2 u.$

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According to the variational iteration method (VIM), the alteration functional for Eq. (3)is given as [1–5]

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\xi)\left[u_n''(\xi) + \omega^2 u_n(\xi) + \tilde{g}(u_n)\right] d\xi, \qquad n = 0, 1, 2, \ldots$$
.….(4)

whereλ is a general Lagrange multiplier, and it can be optimally resolute from the stationary conditionsof Eq. (4)with respect to$u_n$using the variational theory [9–11]. The subscript n represents the $n$thapproximation and$\tilde{g}$is a constrained variation, i.e.,$\delta\tilde{g}$ =0.There are many publications discussing howto recognize the multiplier effectively.

Hereby we will show adifferent approach to the classification of the multiplier. Starting from somepioneering thoughts going back to Abassy, El-Tawil and El-Zoheiry in 2007 [12], Mokhtari and Mohammadi in2009 [13], Hesameddini and Latifizadeh in 2009 [14], the Laplace transform was adopted in the variationaliteration method. Abassy, El-Tawil and El-Zoheiry [12] used Laplace transform in the resultmethod,the variational iteration technique leads to a succession of linear equations, which can be easily solved by theLaplace transform. Mokhtari and Mohammadi [13] found with the intention of the variational iteration algorithm could besimply constructed by the Laplace transform without using the alteration functional (the variational theory)and restricted variations. Hesameddini and Latifizadeh [14] found that Laplace transform could erectiteration algorithms as those by the variational iteration method. When solving a fractional differentialequation, the variational iteration method shows some obvious advantages over others [15–19], and theLaplace transform plays an even more important role in the solution process [20–22]. The present methodofbelow is also applicable for fractal derivative equations [23–27].
Generally the Lagrange multiplier can be expressed in the form [1–5]

$$\lambda = \lambda(t-\xi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

In view of Eq. (5), the correction functional given in Eq. (4)is basically the convolution; hence we canUse the Laplace transform easily.
Applying the Laplace transform on both sides of Eq. (4), the correctionfunctional will be transformed in the following manner

$$\begin{aligned}
\mathrm{L}\left[u_{n+1}(t)\right] &= \mathrm{L}\left[u_n(t)\right] + \mathrm{L}\left[\int_0^t \lambda(t-\xi)\left[u_n''(\xi) + \omega^2 u_n(\xi) + \tilde{g}(u_n)\right] d\xi\right] \\
&= \mathrm{L}\left[u_n(t)\right] + \mathrm{L}\left[\lambda(t) * \left(u_n''(t) + \omega^2 u_n(t) + \tilde{g}(u_n)\right)\right] \\
&= \mathrm{L}\left[u_n(t)\right] + \mathrm{L}\left[\lambda(t)\right]\mathrm{L}\left[u_n''(t) + \omega^2 u_n(t) + \tilde{g}(u_n)\right] \\
&= \mathrm{L}\left[u_n(t)\right] + \mathrm{L}\left[\lambda(t)\right]\left[(s^2+\omega^2)\mathrm{L}\left[u_n(t)\right] - s u_n(0) - u_n'(0) + \mathrm{L}\left[\tilde{g}(u_n)\right]\right]
\end{aligned}$$

……………………………. (6)

The optimal value of λ can be obtained by making Eq. (6)stationary with respect to $u_n(t)$, this requires

$$\frac{\delta}{\delta u_n}\mathrm{L}\left[u_{n+1}(t)\right] = \frac{\delta}{\delta u_n}\mathrm{L}\left[u_n(t)\right] + \frac{\delta}{\delta u_n}\mathrm{L}\left[\lambda(t)\right]\left[(s^2+\omega^2)\mathrm{L}\left[u_n(t)\right] - su_n(0) - u_n'(0) + \mathrm{L}\left[\tilde{g}(u_n)\right]\right]$$

$$= \left\{1 + \mathrm{L}\left[\lambda(t)\right](s^2+\omega^2)\right\}\frac{\delta \mathrm{L}\left[u_n(t)\right]}{\delta u_n} = 0 \quad \text{…………………………………………… (7)}$$

From Eq. (7), we have

$$\mathrm{L}\left[\lambda\right] = -\frac{1}{(s^2+\omega^2)} \quad \text{……………………………………………………..(8)}$$

In the above derivation, we assume that

$$\frac{\delta \mathrm{L}\left[\tilde{g}(u_n)\right]}{\delta u_n} = 0 \quad \text{………………………………………...(9)}$$

The inverse Laplace transform for Eq. (8)results in

$$\lambda(t) = -\frac{1}{\omega}\sin\omega t \quad \text{…………………………………………..(10)}$$

We, therefore, identify the Lagrange multiplier much easier than that by the variational theory.

An example

As an example, we consider the following oscillator [28,29]:

$$(1+\alpha u^2)u'' + \alpha u u'^2 - u(1-u^2) = 0 \quad \text{……………..................... (11)}$$

with initial conditions

$$u(0) = A, \qquad u'(0) = 0 \quad \text{……………………….................................. (12)}$$

This example was solved by the homotopy perturbation method [28]. We write Eq. (11)in the form

$$u'' + \omega^2 u + g(u) = 0 \quad \text{………………………….......................... (13)}$$

Where

$$g(u) = -(1+\omega^2)u + \alpha u^2 u'' + \alpha u u'^2 + u^3$$

We have the following iteration formula

$$\mathrm{L}\left[u_{n+1}(t)\right] = \mathrm{L}\left[u_n(t)\right] - \mathrm{L}\left[\int_0^t \frac{1}{\omega}\sin\omega(t-\xi)\left(u_n''(\xi) + \omega^2 u_n(\xi) + g(u_n)\right)d\xi\right]$$
$$= \mathrm{L}\left[u_n(t)\right] - \frac{1}{\omega}\mathrm{L}\left[\sin\omega t\right]\mathrm{L}\left[u_n''(t) + \omega^2 u_n(t) + g(u_n)\right]$$
$$= \mathrm{L}\left[u_n\right] - \frac{1}{\omega}\mathrm{L}\left[\sin\omega t\right]\mathrm{L}\left[u_n'' - u_n + \alpha u_n^2 u_n'' + \alpha u_n u_n'^2 + u_n^3\right]$$

…………………………………… (14)

Assuming the initial solution is

$$u_0(t) = A\cos\omega t$$ ……………………………... (15)

we have

$$\mathrm{L}\left[u_1(t)\right] = \mathrm{L}\left[A\cos\omega t\right] - \frac{1}{\omega}\mathrm{L}\left[\sin\omega t\right]\mathrm{L}\left[-A\omega^2\cos\omega t - A\cos\omega t - \frac{\alpha A^3\omega^2}{4}(3\cos\omega t + \cos 3\omega t)\right.$$
$$\left.+\frac{\alpha A^3\omega^2}{4}(\cos\omega t - \cos 3\omega t) + \frac{A^3}{4}(3\cos\omega t + \cos 3\omega t)\right]$$
$$= \mathrm{L}\left[A\cos\omega t\right] - \frac{1}{\omega}\left(-A\omega^2 - A + \frac{3}{4}A^3 - \frac{1}{2}\alpha A^3\omega^2\right)\mathrm{L}\left[\sin\omega t\right]\mathrm{L}\left[\cos\omega t\right]$$
$$-\frac{1}{\omega}\left(\frac{1}{4}A^3 - \frac{1}{2}\alpha A^3\omega^2\right)\mathrm{L}\left[\sin\omega t\right]\mathrm{L}\left[\cos 3\omega t\right]$$

………………………………………….. (16)

The inverse Laplace transform on Eq. (16)results in the first order approximate solution:

$$u_1(t) = A\cos\omega t - \frac{1}{\omega}\left(-A\omega^2 - A + \frac{3}{4}A^3 - \frac{1}{2}\alpha A^3\omega^2\right)\left(\frac{1}{2}t\sin\omega t\right)$$
$$-\frac{1}{\omega}\left(\frac{1}{4}A^3 - \frac{1}{2}\alpha A^3\omega^2\right)\left(\frac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right)$$

……...

(17)

No secular-term in Eq. (17)requires that

$$\frac{1}{\omega}\left(-A\omega^2 - A + \frac{3}{4}A^3 - \frac{1}{2}\alpha A^3\omega^2\right) = 0$$

…………………………. (18)

which leads to the following result

$$\omega = \sqrt{\frac{\frac{3}{4}A^2 - 1}{1 + \frac{1}{2}\alpha A^2}}$$ ……………………………………………….. (19)

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Eq. (19)is exactly same as that obtained by the homotopy perturbation method [28] or He’s frequency–amplitude formulation [30].

Discussion and conclusion

In this short paper we apply the Laplace transform to identify easily the Lagrange multiplier. As theLaplace transform is widely known to almost all non-mathematicians, such identification of the Lagrangemultiplier makes the variational iteration method accessible to all researchers who face various nonlinearproblems. The use of the variational iteration process now requires no particular knowledge of elusive calculusof variations. Though this paper gives a basic solution process to a nonlinear oscillator, the method is validfor other nonlinear problems as well.

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